ME469A

Numerical Methods for Fluid Mechanics

Handout #5

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Finite Volume Methods

Last time we introduced the FV method as a discretization technique applied to the integral form of the governing equations

$$\frac{d}{dt}\int_{\Omega_i} \rho\phi dV + \sum_f \int_{S_f} \rho\phi\vec{v}\cdot\hat{n_f} dS = \sum_f \int_{S_f} \Gamma\nabla\phi\cdot\hat{n_f} dS + \int_{\Omega_i} Q_\phi dV$$

Need to use two levels of approximation:

1. the integral is approximated in terms of values defined at one or more points on the cell faces
2. the face values are approximated in terms of the cell center values

Finite Volume Methods

Various integration and interpolation schemes can be used; to achieve accurate results they have to be "synchronized"

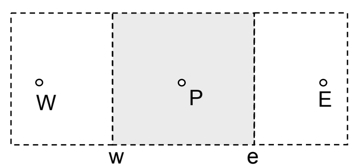
Integration: midpoint rule is typically used
Interpolation: linear or quadratic typically used

We started from the steady form of the conservation eq.

$$\sum_f \int_{S_f} \rho\phi\vec{v}\cdot\hat{n_f}dS = \sum_f \int_{S_f} \Gamma\nabla\phi\cdot\hat{n_f}dS$$

Finite Volume Methods

Conservation equation



$$\int_{S_f} \rho\phi\vec{v}\cdot\hat{n_f}dS = \int_{S_f} \Gamma\nabla\phi\cdot\hat{n_f}dS$$

Convective term

$$\int_{S_f} \rho\phi\vec{v}\cdot\tilde{n}dS = F_e + F_w = \rho v\phi_e A - \rho v\phi_w A$$

Diffusive term

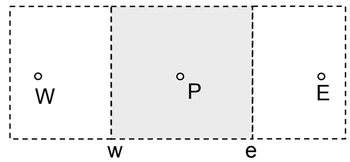
$$-\int_{S_f} \mu\frac{\partial\phi}{\partial x}dS = D_e + D_w = -\mu\left(\frac{\partial\phi}{\partial x}\right)_e A + \mu\left(\frac{\partial\phi}{\partial x}\right)_w A$$

Discrete conservation

$$F_e + D_e + F_w + D_w = 0$$

Linear Interpolation

Assume uniform grid



$$\phi_w = \frac{\phi_i + \phi_{i-1}}{2} \qquad \left(\frac{\partial \phi}{\partial x}\right)_w = \frac{\phi_i - \phi_{i-1}}{\Delta x}$$

$$\phi_e = \frac{\phi_{i+1} + \phi_i}{2} \qquad \left(\frac{\partial \phi}{\partial x}\right)_e = \frac{\phi_{i+1} - \phi_i}{\Delta x}$$

Discrete conservation becomes $F_e + D_e + F_w + D_w = 0$

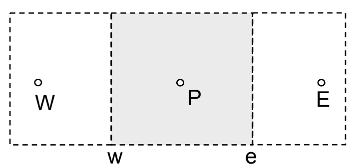
$$\rho v \frac{\phi_{i+1} + \phi_i}{2} A - \mu \frac{\phi_{i+1} - \phi_i}{\Delta x} A - \rho v \frac{\phi_i + \phi_{i-1}}{2} A + \mu \frac{\phi_i - \phi_{i-1}}{\Delta x} A = 0$$

Collecting

$$\left(-\frac{\rho v}{2} - \frac{\mu}{\Delta x}\right) \phi_{i-1} + 2\frac{\mu}{\Delta x}\phi_i + \left(\frac{\rho v}{2} - \frac{\mu}{\Delta x}\right) \phi_{i+1} = 0$$

Linear Interpolation

Discrete Algebraic System



$$A_W \phi_{i-1} + A_P \phi_i + A_E \phi_{i+1} = 0$$

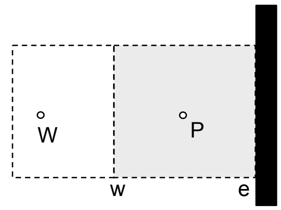
$$\begin{pmatrix} \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \\ & A_W & A_P & A_E & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \phi_i \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} \cdot \\ 0 \\ 0 \\ 0 \\ \cdot \end{pmatrix}$$

Tridiagonal system: how to solve it?
What to do at the boundaries?

Boundary Volumes

Each CV is associated to one equation (one unknown)

For cells on the boundaries the fluxes have to be expressed involving only the data (boundary values) and internal values (one sided extrapolation)



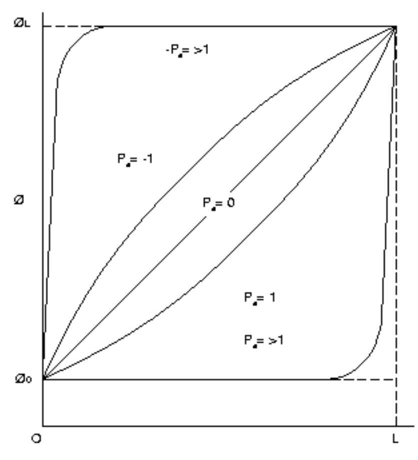
Note: the convective flux can be expressed directly with $\phi_e = \phi_{BC}$ while the diffusive flux requires an hypothesis for the gradient of the solution

$$\begin{pmatrix} \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \\ & A_W & A_P & A_E & \\ & & \cdot & \cdot & \cdot \\ & & & A_W & \tilde{A}_P \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \phi_i \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} \cdot \\ 0 \\ 0 \\ 0 \\ \tilde{A}_E \phi_{BC} \end{pmatrix}$$

Exact Solution

This (atypical) problem has an exact solution is Dirichlet conditions are applied to x=0 (ϕ_0) and x=L (ϕ_L)

$$\frac{\phi(x) - \phi_0}{\phi_L - \phi_0} = \frac{\exp(Pe\frac{x}{L}) - 1}{\exp(Pe) - 1}$$

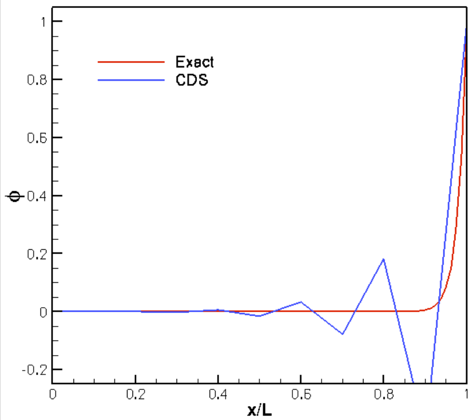


The Peclet number is

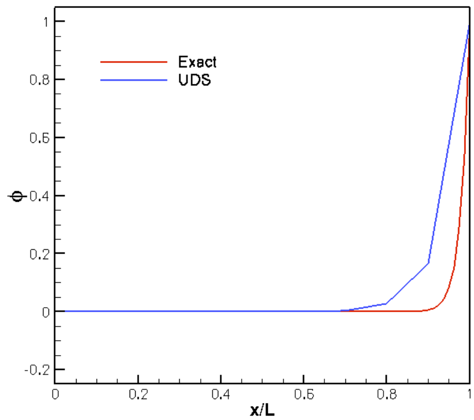
$$Pe = \frac{\rho u L}{\Gamma}$$

FV Solutions

Computational grid: 11 equal volumes
Pe = 50



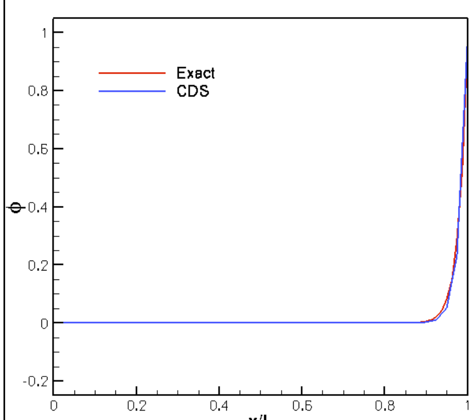
Linear interpolation (CDS)



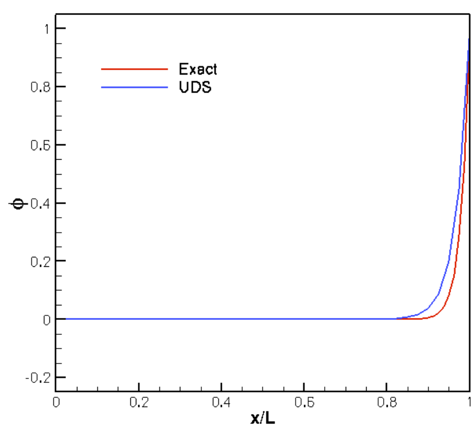
Upwind interpolation (UDS)

FV Solutions

Computational grid: 41 equal volumes
Pe = 50



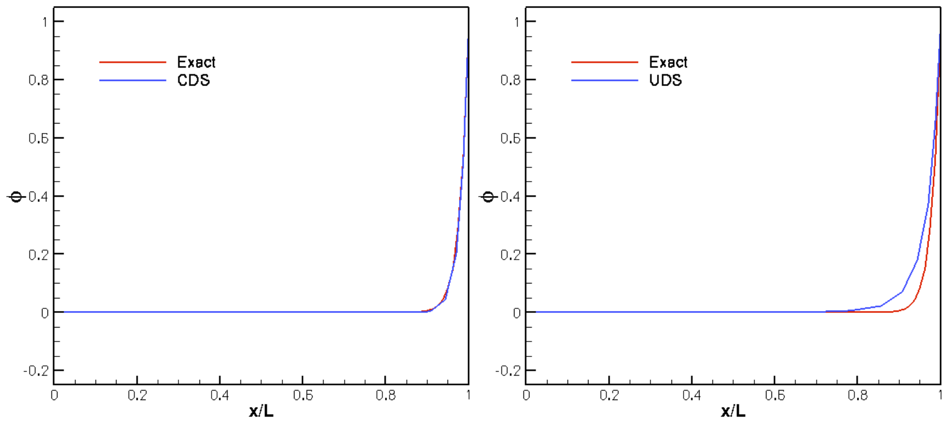
Linear interpolation (CDS)



Upwind interpolation (UDS)

FV Solutions

Computational grid: 11 non-uniform volumes (clustered at x=1)
Pe = 50



Linear interpolation (CDS) Upwind interpolation (UDS)

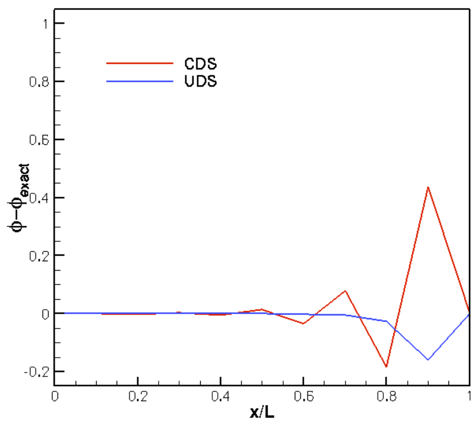
FV Solutions

Observations

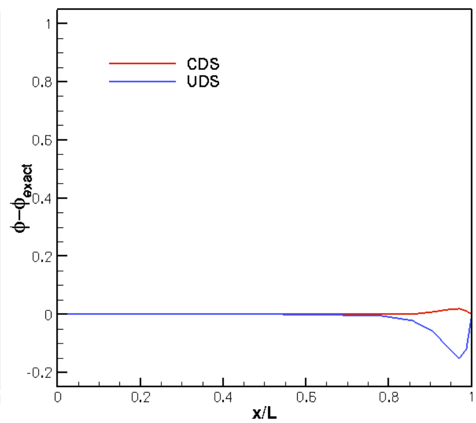
- Upwind differencing introduces considerable amount of "artificial" diffusion – even on coarse of non-uniform meshes
- Central differencing introduces oscillation in the solution (the solution is not bounded by the boundary values) – upwind differencing does NOT
- The oscillations in the CDS disappear on fine grids on on non-uniform grids

FV Solutions

Error distribution in the domain (Pe=50)



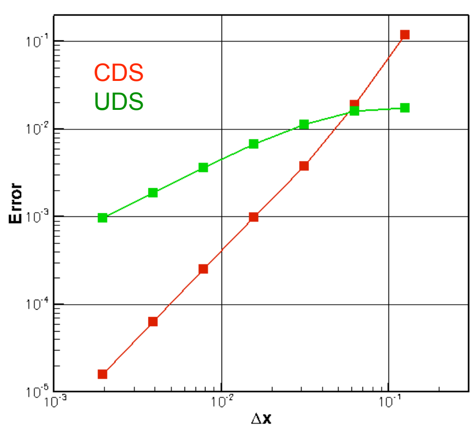
Uniform grid (N=11)



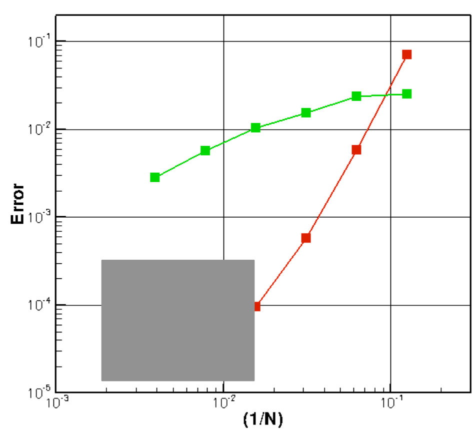
Non uniform grid (N=11)

FV Solutions

Convergence of the truncation error $\epsilon = \frac{1}{N}\sum_i |\phi_i^{exact} - \phi_i|$



Uniform grid (Δx=1/N)

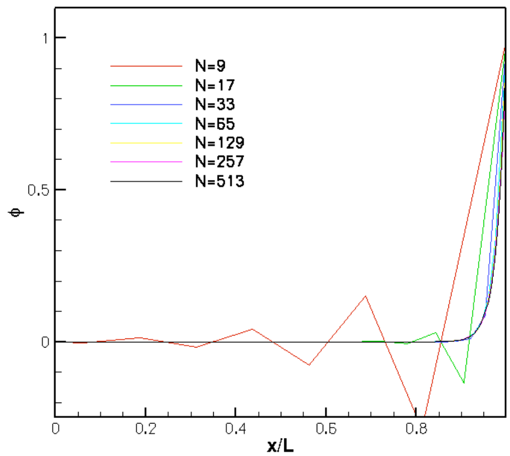


Non uniform grid

FV Solutions

- In spite of the oscillatory nature of the error, the CDS scheme yields second order results, while the UDS only first order – as expected (verification)

- Why is the solution oscillatory?
- What changes with grid refinement?



Bounds on the solutions

The conservation principle governing this problem implies that the solution is bounded $\phi_0<\phi<\phi_L$

This is important because oscillations might cause unphysical behavior: if ϕ represents the concentration of a pollutant (or the temperature)

Boundedness is very difficult to guarantee, some results are available for first order discretization schemes. High order schemes tend to generate wiggles in the region where the solution gradients are high

Bounds on the solutions

A condition for boundedness can be derived from the system of equations

$$A_W\phi_{i-1} + A_P\phi_i + A_E\phi_{i+1} = 0$$

If A_E and A_W have the same sign (and given A_P=-A_E-A_W) the solution is bounded

Given the assumption in the previous example:

$$A_W = \left(-\frac{\rho v}{2} - \frac{\mu}{\Delta x}\right) \leq 0$$

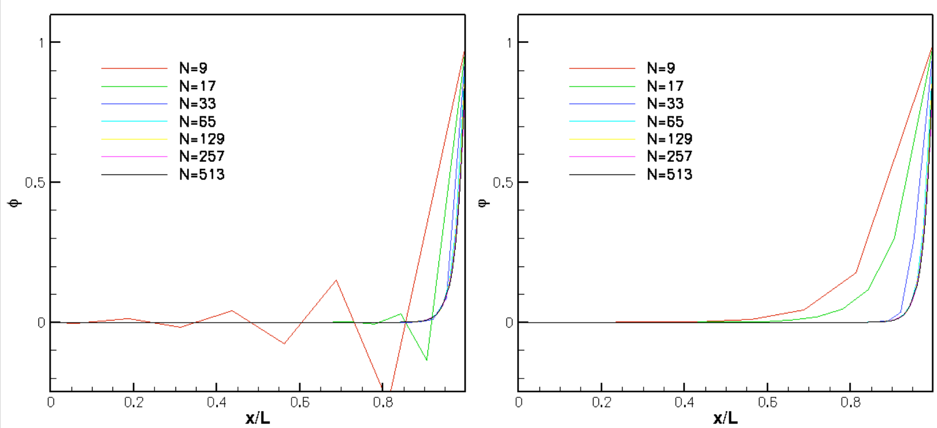
$$A_E = \left(\frac{\rho v}{2} - \frac{\mu}{\Delta x}\right) \leq 0 \quad \text{if} \quad \frac{\rho v \Delta x}{\mu} = Pe_{\Delta x} \leq 2$$

The CDS solution is only bounded if the cell Peclet < 2

Exercise: Verify that the UDS scheme is ALWAYS bounded!

Bounds on the solutions

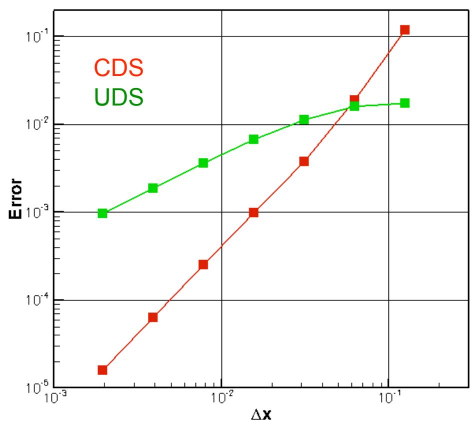
This "limiter" is the basis for the hybrid scheme, which uses CDS/UDS discretization according to cell Peclet number

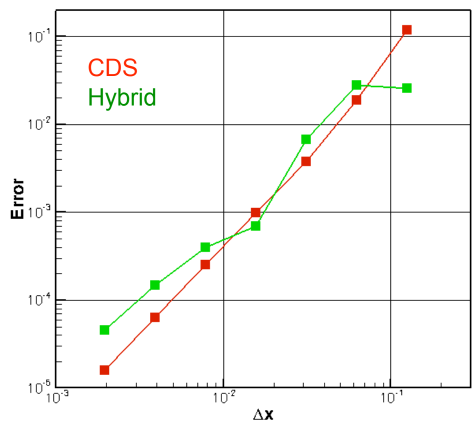


CDS Hybrid

FV Solutions

Convergence of the truncation error

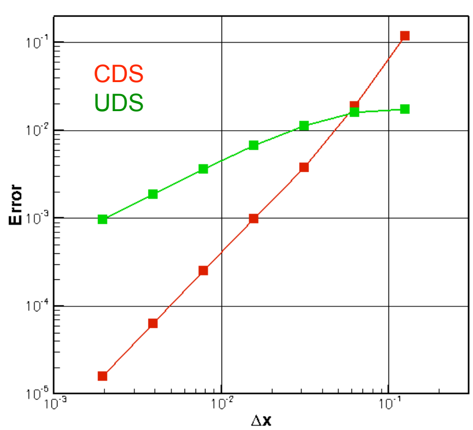


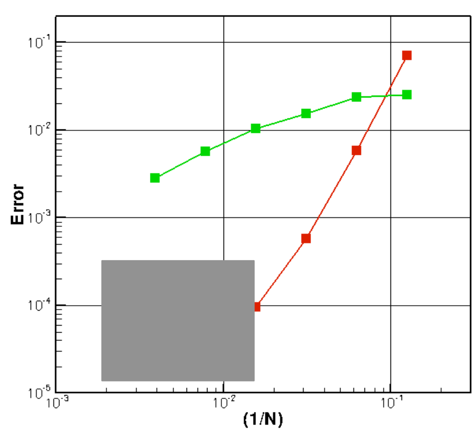


Uniform grid (Δx=1/N)

FV Solutions

Convergence of the truncation error



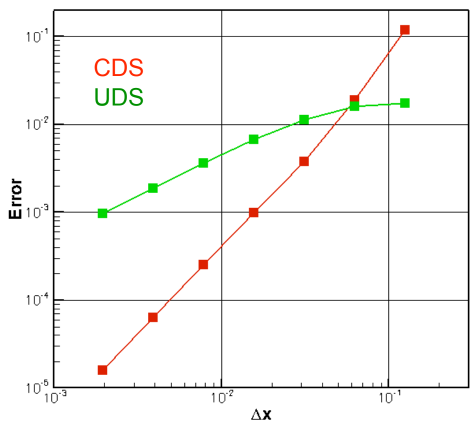


Uniform grid (Δx=1/N)

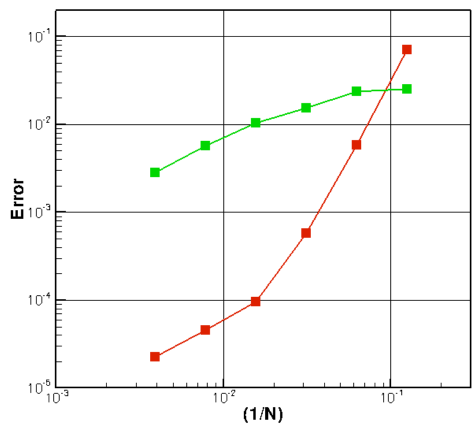
Non uniform grid

FV Solutions

Convergence of the truncation error



Uniform grid (Δx=1/N)

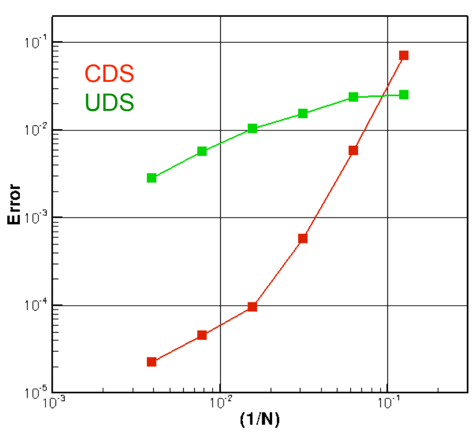


Non uniform grid

FV Solutions

Non-uniform grids generated with

```
DX=(XMAX-XMIN)*(1.-EX)/
                  (1.-EX**(N-1))
X(1)=XMIN
DO I=2,N
    X(I)=X(I-1)+DX
    DX=DX*EX
END DO
```

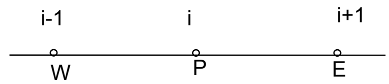


Non uniform grid sequence

	Δx_N	1/N
1	0.08398057	0.1111111
2	0.02527204	0.0588235
3	0.00399518	0.0303030
4	0.00010743	0.0153846
5	1.51316E-7	0.0077519
6	2.1460E-13	0.0038910

Finite Difference Method

Differential equation



$$\rho v \frac{\partial \phi}{\partial x} = \Gamma \frac{\partial^2 \phi}{\partial x^2}$$

Use central differencing

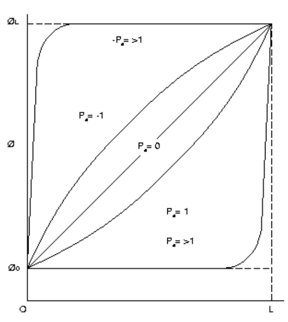
$$\rho v \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} = \Gamma \frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{\Delta x^2}$$

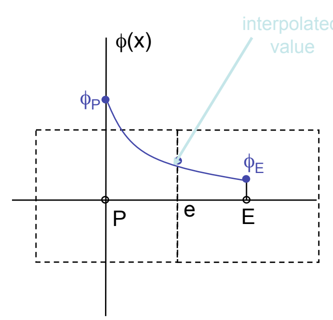
Collecting $$A_W \phi_{i-1} + A_P \phi_i + A_E \phi_{i+1} = 0$$

We obtain exactly the FV system,
Remarks: extensions to multi-dimensions and variable coefficient are not equivalent

Exponential Interpolation Scheme

The exact solution can be used to develop an interpolation scheme





$$\frac{\phi(x) - \phi_0}{\phi_L - \phi_0} = \frac{\exp(Pe\frac{x}{L}) - 1}{\exp(Pe) - 1}$$

$$\frac{\phi_e - \phi_P}{\phi_E - \phi_P} = \frac{\exp\left[Pe x_e/(x_E - x_P)\right] - 1}{\text{exp(Pe)-1}}$$

Exponential Interpolation Scheme

The east fluxes are

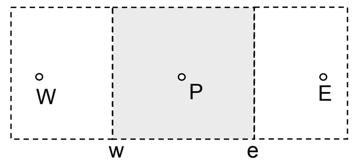
$$\phi_e = \phi_i(1-\lambda_e) + \phi_{i+1}\lambda_e$$

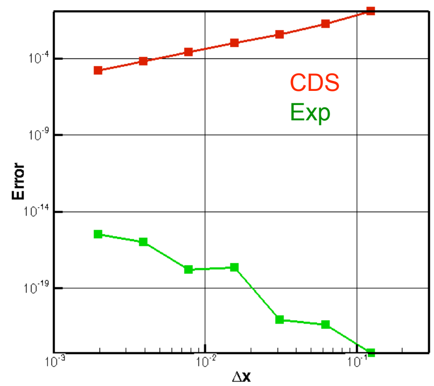
$$\left(\frac{\partial \phi}{\partial x}\right)_e = \frac{\phi_{i+1}-\phi_i}{\Delta x}\lambda_e \mathrm{Pe}$$

with $\lambda_e = \dfrac{\exp(Pe/2)-1}{\exp(Pe)-1}$

Similarly for the west fluxes

Should be VERY accurate





Exponential Interpolation Scheme

Why this is not a useful scheme?

Physics-based but related to a very atypical situation (balance of convection and diffusion in streamwise direction)

Exact solution ONLY valid in 1D, steady with no source terms!

Reconstruction depends on the local Peclet number:
1. Convection flux depends on the diffusion coefficient
2. Diffusion flux depends on the convection velocity

Expensive to compute EXP functions

So far…

Confirmed convergence rate for UDS and CDS schemes

UDS introduces “false”/”artificial” dissipation but it is bounded
CDS is more accurate but not bounded

A boundedness condition can be derived based on the local Peclet number. As a consequence a hybrid scheme can be derived

“Exact” schemes for 1D steady convection/diffusion equation can be derived

Non-asymptotic convergence MUST be explained

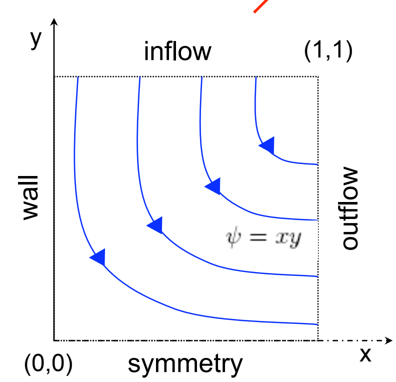
Extension to Multi-Dimensions

$$\frac{d}{dt}\int_{\Omega}\rho\phi dV + \int_S \rho\phi\vec{v}\cdot\hat{n}dS = \int_S \Gamma\nabla\phi\cdot\hat{n}dS + \int_{\Omega} Q_{\phi}dV$$

Assume the velocity field

$$u = x$$

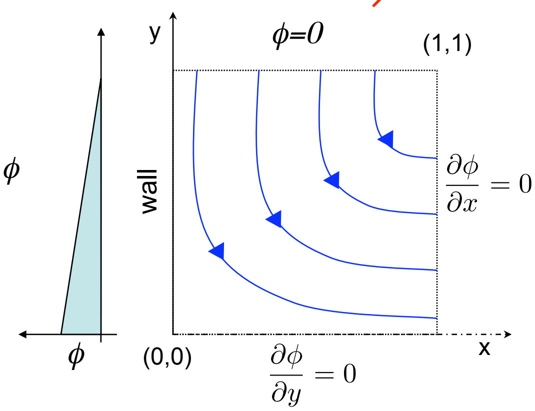
$$v = -y$$



Extension to Multi-Dimensions

$$\frac{d}{dt}\int_{\Omega}\rho\phi dV + \int_S \rho\phi\vec{v}\cdot\hat{n}dS = \int_S \Gamma\nabla\phi\cdot\hat{n}dS + \int_{\Omega} Q_\phi dV$$

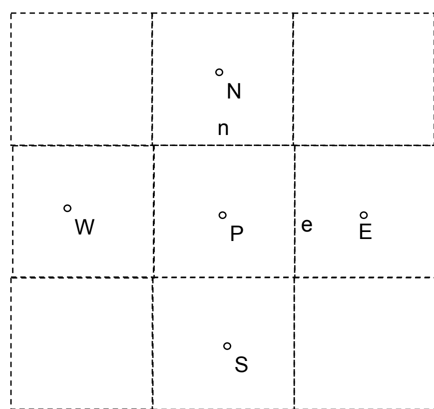
Specify the boundary conditions for the scalar ϕ as a mix of Dirichlet and Neumann



Extension to Multi-Dimensions

Need to consider now 8 flux contributions per cell
(4 convective + 4 diffusive)

$$\sum_{f=1}^{4}\int_{S_f}\rho\phi\vec{v}\cdot\hat{n_f}dS = \sum_{f=1}^{4}\int_{S_f}\Gamma\nabla\phi\cdot\hat{n_f}dS$$



Diffusive fluxes are easy

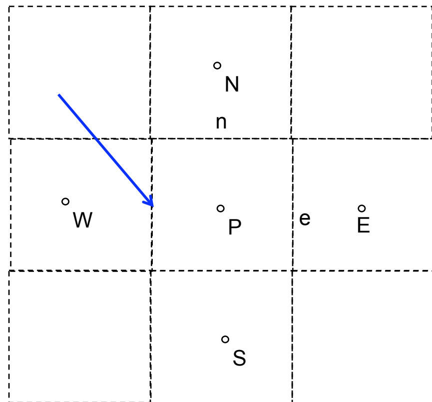
Linear interpolation involving the face-neighbors leads to second order accuracy

Extension to Multi-Dimensions

Convective fluxes are more difficult

Need the convection velocity

$$(\vec{v} \cdot \hat{n_f})_e = u$$
$$(\vec{v} \cdot \hat{n_f})_n = v$$



x in this case is not the streamwise direction, so how do we define upwinding?

It is not a combination of 1D problems!

Multi-D Convection Schemes

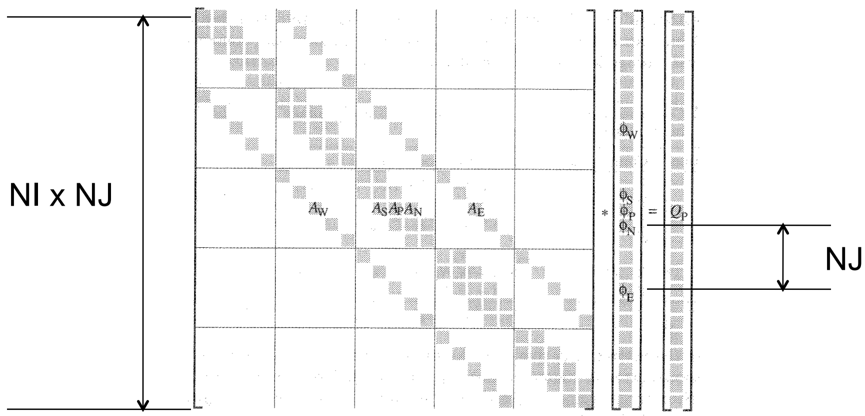
QUICKEST - Quadratic Upstream Interpolation for Convective Kinematics with Estimated Streaming Terms
AQUATIC - Adjusted Quadratic Upstream Algorithm for Transient Incompressible Convection
ULTRA-SHARP : Universal Limiter for Thight Resolution and Accuracy in combination with the Simple High-Accuracy Resolution Program (also ULTRA-QUICK)
UTOPIA - Uniformly Third Order Polynomial Interpolation Algorithm
NIRVANA - Non-oscilatory Integrally Reconstructed Volume-Avaraged Numerical Advection scheme
ENIGMATIC - Extended Numerical Integration for Genuinely Multidimensional Advective Transport Insuring Conservation6
MACHO : Multidimensional Advective - Conservative Hybrid Operator
COSMIC : Conservative Operator Splitting for Multidimensions with Internal Constancy
EXQUISITE - Exponential or Quadratic Upstream Interpolation for Solution of the Incompressible Transport Equation
LODA - Local Oscillation-Damping Algorithm
MSOU - Monotonic Second Order Upwind Differencing Scheme

http://www.cfd-online.com/Wiki/Approximation_Schemes_for_convective_term

Extension to Multi-Dimensions

Using linear reconstruction we obtain a 5-point scheme

$$A_S \phi_{i,j-1} + A_W \phi_{i-1,j} + A_P \phi_{i,j} + A_E \phi_{i+1,j} + A_N \phi_{i,j+1} = 0$$



Error Analysis

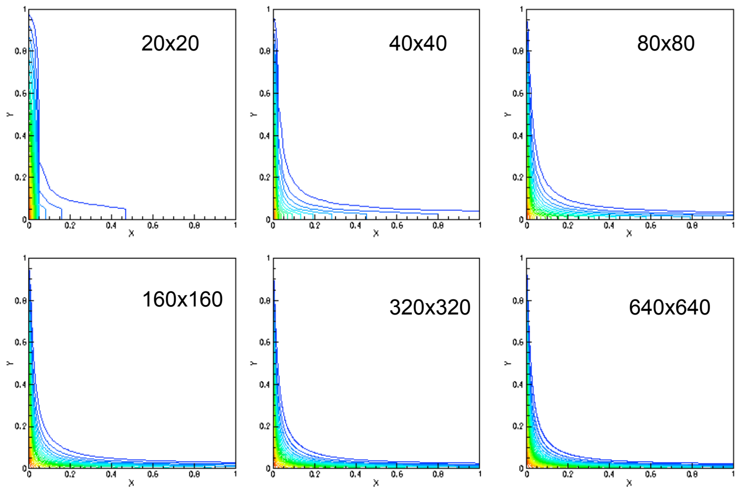
As in the 1D case, we would like to verify the accuracy of the discretization schemes

Do not have an EXACT solution!

Solve on a sequence of uniform meshes (using psc.f) and use the finest grid as a reference

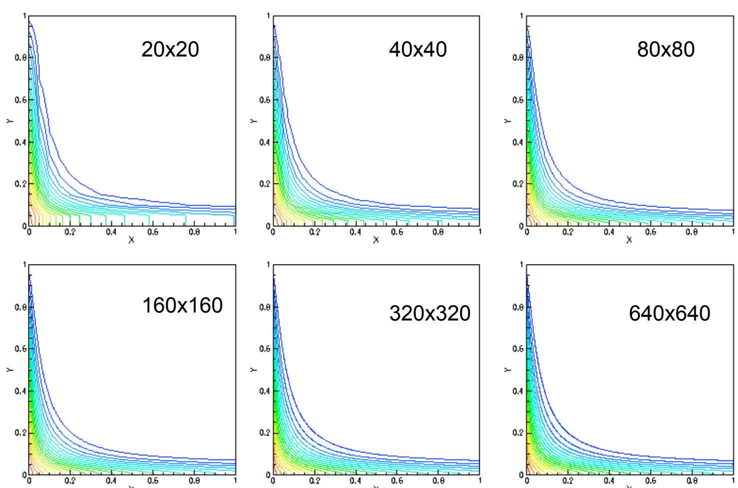
Upwind - Γ=0.01

Isolines of ϕ



Central - Γ=0.1

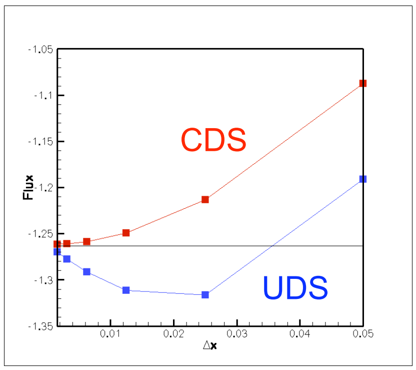
Isolines of ϕ



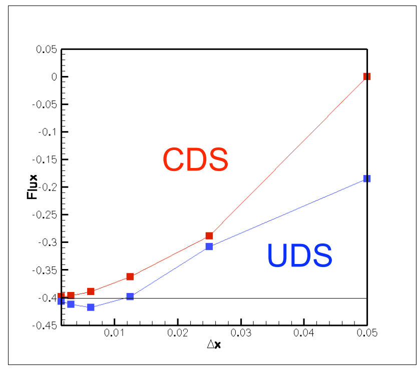
Monitor Grid Convergence

Flux of ϕ at x=0

Γ=0.1



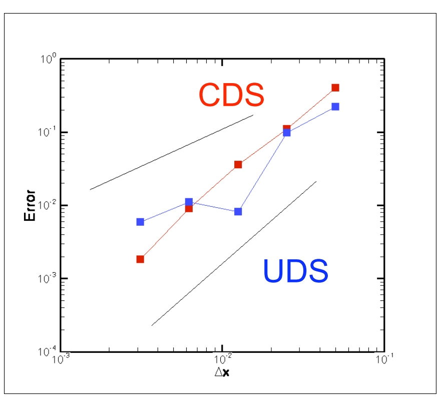
Γ=0.01



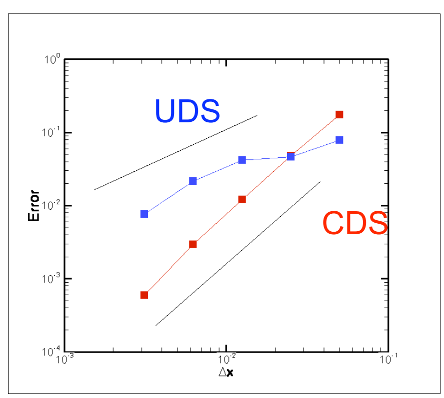
Computing Errors

Flux of ϕ at x=0 - Using Fine Grid Solution as Reference

Γ=0.1

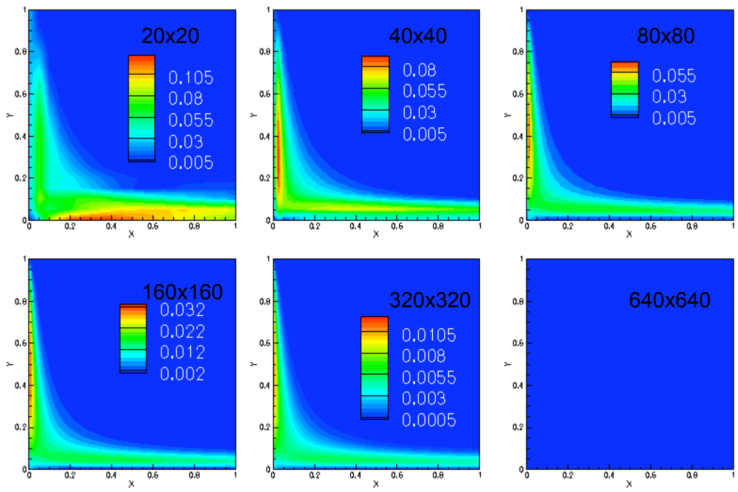


Γ=0.01



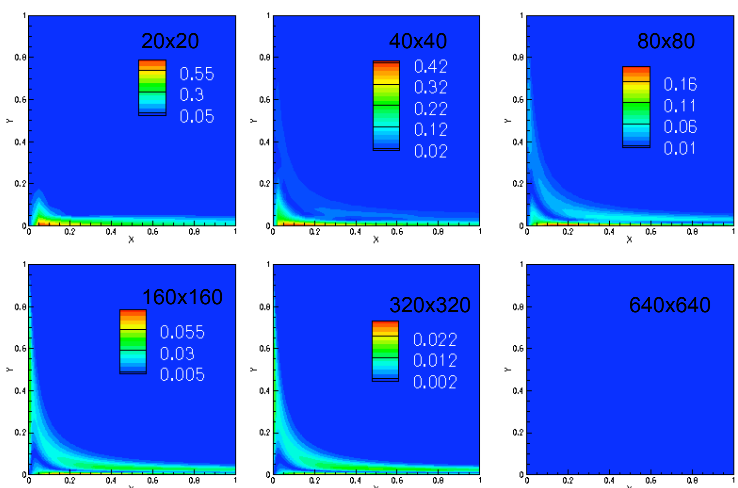
Computing Errors

Isolines of Error in ϕ - CDS - Γ = 0.1

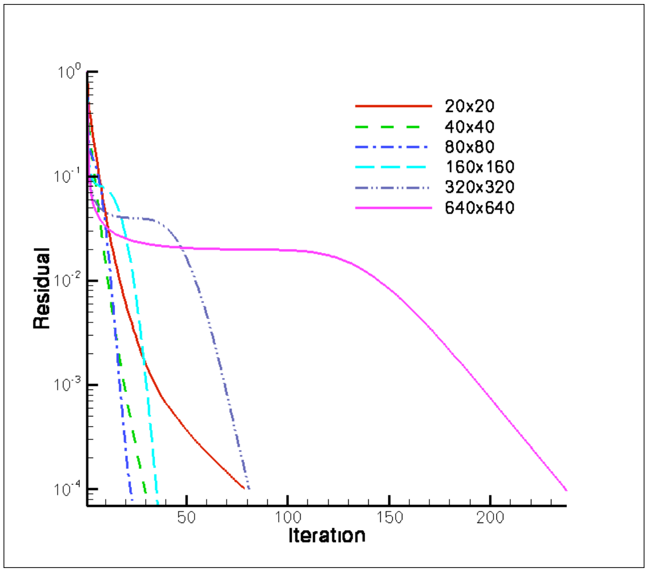


Computing Errors

Isolines of Error in ϕ - UDS - Γ = 0.01



Computational Cost



CDS