Solution methods for the Unsteady Incompressible Navier-Stokes Equations
Unsteady flows

The algorithms we introduced so far are time-marching:
From an initial condition they iterate until a steady-state is reached
The “time”-evolution of the solution is NOT accurate

Typical Implicit Time-Accurate Scheme

\[ \frac{\partial \phi}{\partial t} = F(\phi) \quad \text{Unsteady transport equation} \]
\[ \frac{\phi^{m+1} - \phi^m}{\Delta t} = F(\phi^{m+1}) \quad \text{1st order time integration} \]
\[ \frac{3\phi^{m+1} - 4\phi^m + \phi^{m-1}}{2\Delta t} = F(\phi^{m+1}) \quad \text{2nd order time integration} \]
Unsteady flows – Implicit Pressure-based

Generic Transport Equation

\[ \int_V \frac{\partial \rho \phi}{\partial t} dV + \oint \rho \phi \vec{v} \cdot d\vec{A} = \oint \Gamma \phi \nabla \phi \cdot d\vec{A} + \int_V S_{\phi} dV \]

Fully Implicit Discretization

\[ \int_V \frac{\partial \rho \phi}{\partial t} dV + \oint \rho^{n+1} \phi^{n+1} \vec{v}^{n+1} \cdot d\vec{A} = \oint \Gamma^{n+1} \nabla \phi^{n+1} \cdot d\vec{A} + \int_V S_{\phi}^{n+1} dV \]

Frozen Flux Formulation

\[ \oint \rho \phi \vec{v} \cdot d\vec{A} = \oint \rho^n \phi^{n+1} \vec{v}^n \cdot d\vec{A} \]
Unsteady flows – Pressure-based Methods

Iterative

Non-Iterative
Unsteady flows – Set Up

Define → Models → Solver

Solve → Iterate

Outer iteration

Inner iteration
Unsteady Flow – Impulsive start-up of a plate

Again an analytical solution of the Navier-Stokes equations can be derived:

Solution in the form $u = u(y, t)$

The only force acting is the viscous drag on the wall

Navier-Stokes equations

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

Velocity distribution

$$\frac{u(y, t)}{V_{\text{wall}}} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{y/2\sqrt{\nu t}} e^{-\chi^2} d\chi = 1 - \text{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

Wall shear stress

$$\tau_{\text{wall}} = -\mu \left( \frac{\partial u}{\partial y} \right)_{y=0} = \mu \left( \frac{V_{\text{wall}}}{\sqrt{\pi \nu t}} \right)$$
Unsteady Flow – Impulsive start-up of a plate

Problem set-up

Material Properties:
\( \rho = 1 \text{kg/m}^3 \)
\( \mu = 0.1 \text{kg/ms} \)

Reynolds number:
\( \text{Re} = \rho V_{\text{wall}} L / \mu \)
\( V_{\text{wall}} = 5.605 \)
\( L = \mu V_{\text{wall}} / \tau_{\text{wall}} \)

Boundary Conditions:
Slip wall (\( u = V_{\text{wall}} \)) on bottom
No-slip wall (top)
Periodicity \( \Delta p = 0 \)

Initial Conditions:
\( u = v = p = 0 \)

Exact Solution
\( \tau_{\text{wall}} = 1 \) @ \( t = 1 \)
\( H/L \sim 10 \)

Solver Set-Up

Segregated Solver

Discretization:
2\(^{nd}\) order upwind
SIMPLE

Multigrid
V-Cycle
Unsteady Flow – Impulsive start-up of a plate

The error CAN be computed with reference to the exact solution. In this case the computed wall shear stress is plotted.

Time discretization

1\textsuperscript{st} order  2\textsuperscript{nd} order

Influence of the BCs

Nominal 1\textsuperscript{st} order accuracy

Nominal 2\textsuperscript{nd} order accuracy

This test-case is available on the class web site.
Unsteady Flow – Density based formulation

Vector form of the (compressible) NS equations

\[
\frac{\partial}{\partial t} \int_V W \, dV + \oint [F - G] \cdot dA = \int_V H \, dV
\]

\[
W = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix}, \quad F = \begin{pmatrix} \rho v \\ \rho v u + p_i \\ \rho v v + p_j \\ \rho v w + p_k \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ \tau_{xi} \\ \tau_{yi} \\ \tau_{zi} \\ \tau_{ij} v_j + q \end{pmatrix}
\]

Change of variables

\[
\frac{\partial W}{\partial Q} \frac{\partial}{\partial t} \int_V Q \, dV + \oint [F - G] \cdot dA = \int_V H \, dV
\]

\[
\tilde{Q} = [p, \vec{V}, T]^T
\]

Preconditioning

\[
\Gamma \frac{\partial}{\partial t} \int_V Q \, dV + \oint [F - G] \cdot dA = \int_V H \, dV
\]
Unsteady Flow – Density based formulation

For time-accurate simulations the preconditioning cannot be used (it alters the propagation speed of the acoustic signals)

Time integration:

Implicit - n is the time step loop, k is the inner iteration loop

$\Delta t$ determines the time accuracy, $\Delta \tau$ is a pseudo-time step determined by stability conditions (a CFL number)

$$\left[ \frac{\Gamma}{\Delta \tau} + \frac{\epsilon_0}{\Delta t} \frac{\partial W}{\partial Q} \right] \Delta Q^{k+1} + \frac{1}{V} \oint [F - G] \cdot dA$$

$$= H + \frac{1}{\Delta t} \left( \epsilon_0 W^k - \epsilon_1 W^n + \epsilon_2 W^{n-1} \right)$$

$\epsilon_0 = \epsilon_1 = 1/2, \epsilon_2 = 0$ First order

$\epsilon_0 = 3/2, \epsilon_1 = 2, \epsilon_2 = 1/2$ Second order
Reynolds-Averaged Navier-Stokes Equations

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \frac{\mu}{\rho} \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left( -u_i' u_j' \right)
\]

Define Reynolds-averaged quantities

\[
u_i(x_k, t) = U_i(x_k) + u'(x_k, t)
\]

\[
U_i(x_k) = \lim_{T \to \infty} \frac{1}{T} \int_0^T u(x_k, t) dt
\]

Substitute and average:

\[
\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \frac{\mu}{\rho} \frac{\partial U_i}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left( -u_i' u_j' \right)
\]

\[
R_{ij} = -u_i' u_j'
\]

Closure problem
Unsteady RANS?

Every turbulent flow is unsteady BUT not all the unsteadiness is *turbulence*!

RANS averaging based on time average can be applied only to “statistically” steady flows. What if flow has a large scale periodicity (vortex shedding)?

We can define the Reynolds-Averaging procedure in terms of *Ensemble Average*:

\[
U_i(x_k, t) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u_i^{(n)}(x_k, t)
\]

\[
u_i(x_k, t) = U_i(x_k, t) + u'(x_k, t)
\]
Turbulent Vortex Shedding
Turbulent Vortex Shedding