Solution methods for the Unsteady Incompressible Navier-Stokes Equations

Unsteady flows

The algorithms we introduced so far are time-marching:

From an initial condition they iterate until a steady-state is reached

The "time"-evolution of the solution is NOT accurate

Typical Implicit Time-Accurate Scheme

$$\frac{\partial \phi}{\partial t} = F(\phi)$$

Unsteady transport equation

$$\frac{\phi^{m+1} - \phi^m}{\Delta t} = F(\phi^{m+1})$$

1st order time integration

$$\frac{3\phi^{m+1} - 4\phi^m + \phi^{m-1}}{2\Delta t} = F(\phi^{m+1})$$

2nd order time integration

Unsteady flows – Implicit Pressure-based

Generic Transport Equation

$$\int_V \frac{\partial \rho\phi}{\partial t}\, dV + \oint \rho\phi\, \vec{v} \cdot d\vec{A} = \oint \Gamma_\phi \nabla\phi \cdot d\vec{A} + \int_V S_\phi\, dV$$

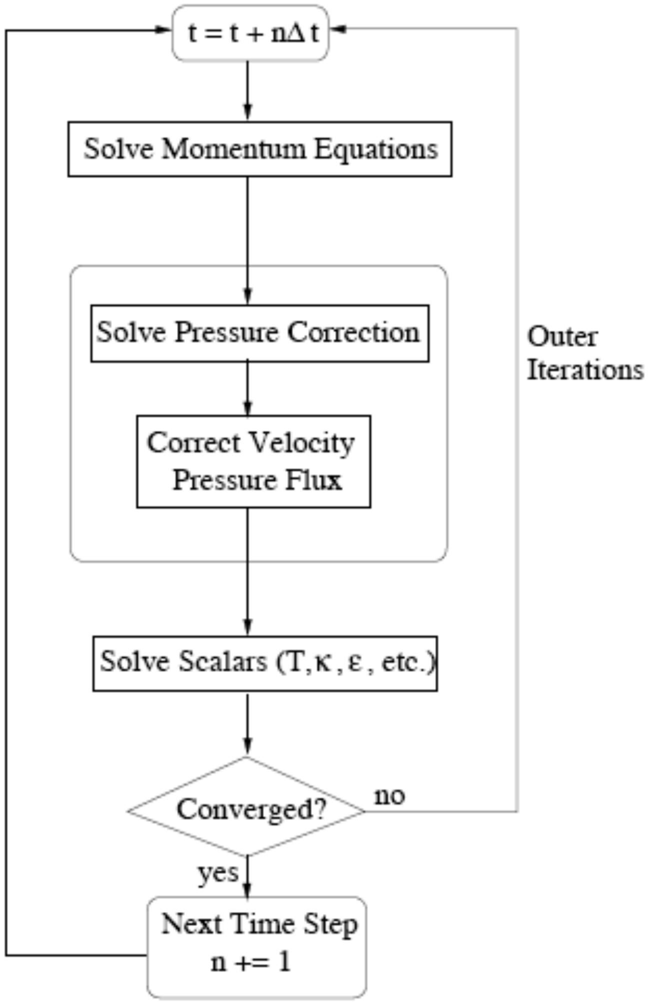
Fully Implicit Discretization

$$\int_V \frac{\partial \rho\phi}{\partial t}\, dV + \oint \rho^{n+1}\phi^{n+1}\, \vec{v}^{n+1} \cdot d\vec{A} = \oint {\Gamma_\phi}^{n+1} \nabla\phi^{n+1} \cdot d\vec{A} + \int_V {S_\phi}^{n+1}\, dV$$

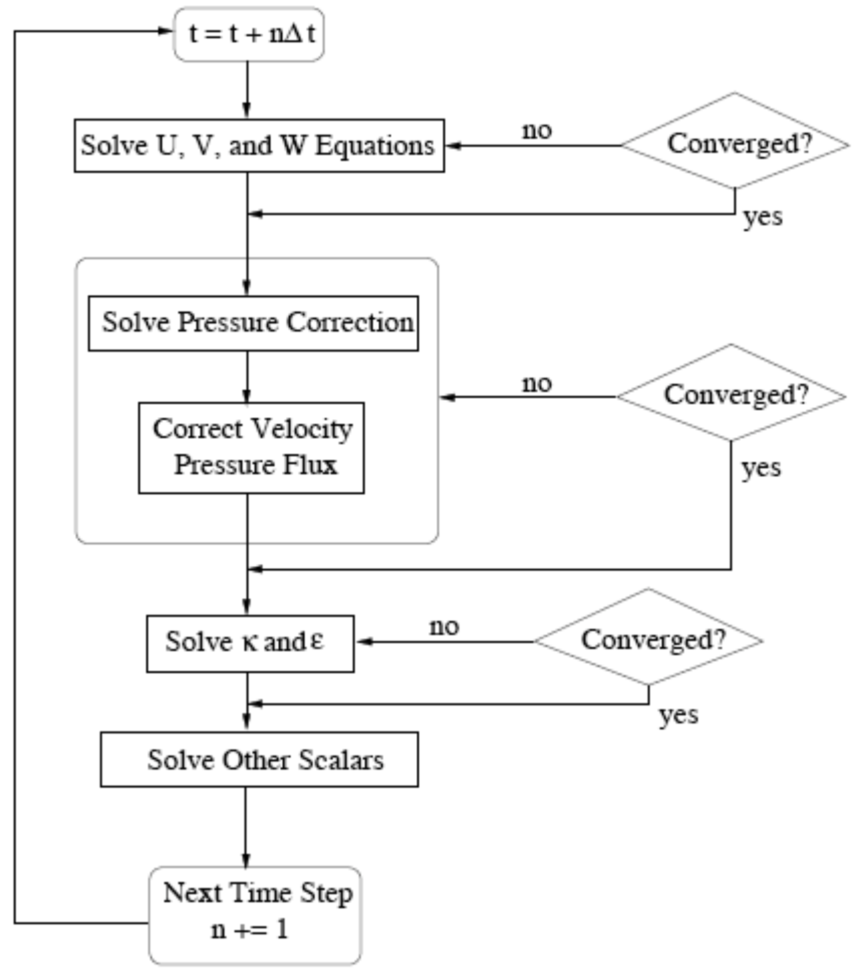
Frozen Flux Formulation

$$\oint \rho\phi\, \vec{v} \cdot d\vec{A} = \oint \rho^{n}\phi^{n+1}\, \vec{v}^{n} \cdot d\vec{A}$$

Unsteady flows – Pressure-based Methods



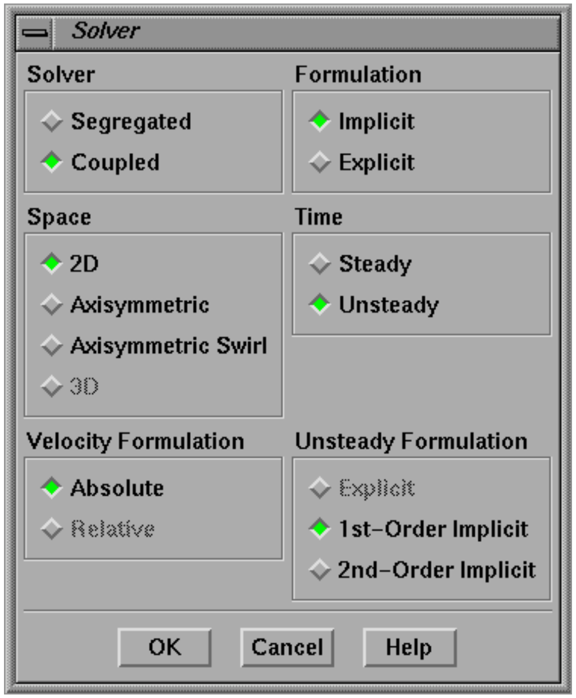
Iterative



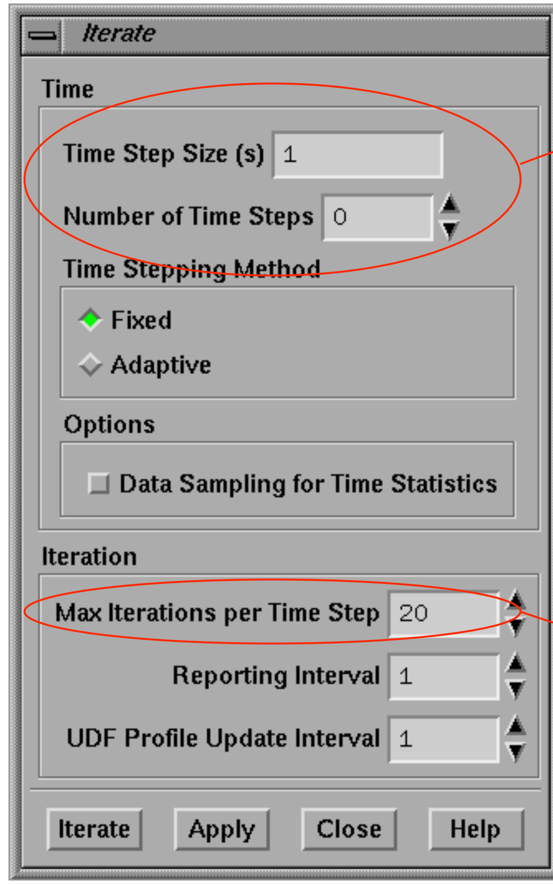
Non-Iterative

Unsteady flows – Set Up

Define → Models → Solver



Solve → Iterate

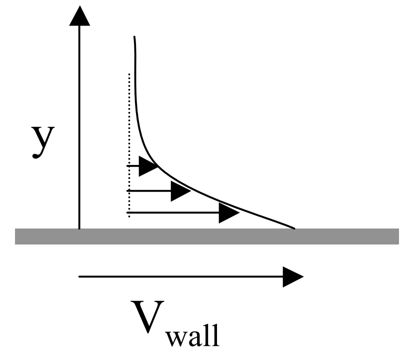


Outer iteration

Inner iteration

Unsteady Flow – Impulsive start-up of a plate

Again an analytical solution of the Navier-Stokes equations can be derived:



$$u(y=0,t) = V_{wall}; u(y=\infty,t) = 0$$

Solution in the form *u=u(y,t)*

The only force acting is the viscous drag on the wall

Navier-Stokes equations

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

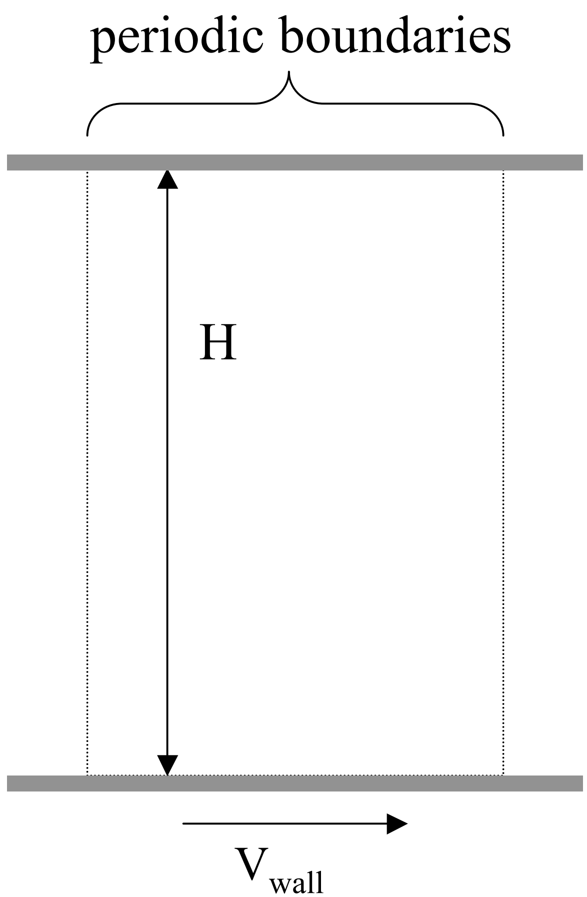
Velocity distribution

$$\frac{u(y,t)}{V_{wall}} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{y/2\sqrt{\nu t}} e^{-\chi^2} d\chi = 1 - \mathrm{erf}\left(y/2\sqrt{\nu t}\right)$$

Wall shear stress

$$\tau_{wall} = -\mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \left(\frac{V_{wall}}{\sqrt{\pi \nu t}}\right)$$

Unsteady Flow – Impulsive start-up of a plate



Problem set-up

Material Properties:

$\rho = 1 kg/m^3$

$\mu = 0.1 kg/ms$

Reynolds number:

$Re = \rho V_{wall} L/\mu$

$V_{wall} = 5.605$

$L = \mu V_{wall}/\tau_{wall}$

Boundary Conditions:

Slip wall ($u = V_{wall}$) on bottom

No-slip wall (top)

Periodicity $\Delta p = 0$

Initial Conditions:

$u = v = p = 0$

Exact Solution

$\tau_{wall} = 1$ @ $t = 1$ $\quad$ $H/L \sim 10$

Solver Set-Up

Segregated Solver

Discretization:

2nd order upwind

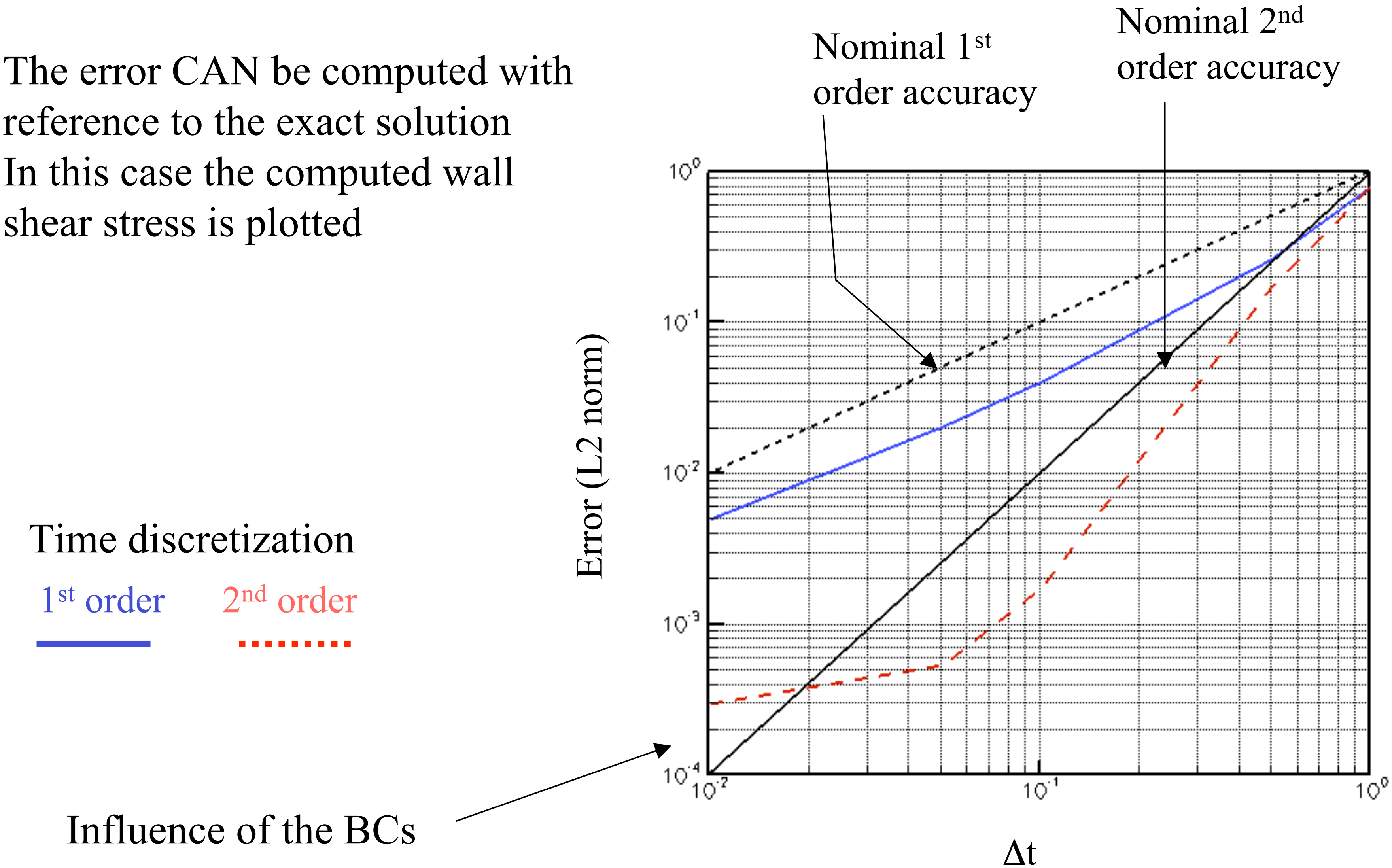
SIMPLE

Multigrid

V-Cycle

Unsteady Flow – Impulsive start-up of a plate

The error CAN be computed with reference to the exact solution
In this case the computed wall shear stress is plotted



This test-case is available on the class web site

Unsteady Flow – Density based formulation

Vector form of the (compressible) NS equations

$$\frac{\partial}{\partial t}\int_V \mathbf{W}\,dV + \oint [\mathbf{F}-\mathbf{G}]\cdot d\mathbf{A} = \int_V \mathbf{H}\,dV$$

$$\mathbf{W} = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{Bmatrix},\ \mathbf{F} = \begin{Bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}u + p\hat{\mathbf{i}} \\ \rho\mathbf{v}v + p\hat{\mathbf{j}} \\ \rho\mathbf{v}w + p\hat{\mathbf{k}} \\ \rho\mathbf{v}E + p\mathbf{v} \end{Bmatrix},\ \mathbf{G} = \begin{Bmatrix} 0 \\ \tau_{xi} \\ \tau_{yi} \\ \tau_{zi} \\ \tau_{ij}v_j + \mathbf{q} \end{Bmatrix}$$

Change of variables

$$\frac{\partial \mathbf{W}}{\partial \mathbf{Q}}\frac{\partial}{\partial t}\int_V \mathbf{Q}\,dV + \oint [\mathbf{F}-\mathbf{G}]\cdot d\mathbf{A} = \int_V \mathbf{H}\,dV$$

$$\vec{Q} = \left[p, \vec{V}, T\right]^T$$

Preconditioning

$$\Gamma\frac{\partial}{\partial t}\int_V \mathbf{Q}\,dV + \oint [\mathbf{F}-\mathbf{G}]\cdot d\mathbf{A} = \int_V \mathbf{H}\,dV$$

Unsteady Flow – Density based formulation

For time-accurate simulations the preconditioning cannot be used (it alters the propagation speed of the acoustic signals)

Time integration:

Implicit - n is the time step loop, k is the inner iteration loop

Δt determines the time accuracy, $\Delta\tau$ is a pseudo-time step determined by stability conditions (a CFL number)

$$\left[\frac{\Gamma}{\Delta\tau} + \frac{\epsilon_0}{\Delta t}\frac{\partial \mathbf{W}}{\partial \mathbf{Q}}\right]\Delta\mathbf{Q}^{k+1} + \frac{1}{V}\oint[\mathbf{F}-\mathbf{G}]\cdot d\mathbf{A} = \mathbf{H} + \frac{1}{\Delta t}\left(\epsilon_0\mathbf{W}^k - \epsilon_1\mathbf{W}^n + \epsilon_2\mathbf{W}^{n-1}\right)$$

$\epsilon_0 = \epsilon_1 = 1/2, \epsilon_2 = 0$ First order

$\epsilon_0 = 3/2, \epsilon_1 = 2, \epsilon_2 = 1/2$ Second order

Reynolds-Averaged Navier-Stokes Equations

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\frac{\mu}{\rho}\frac{\partial u_i}{\partial x_j}\right)$$

Define Reynolds-averaged quantities

$$u_i(x_k,t) = U_i(x_k) + u'(x_k,t)$$

$$U_i(x_k) = \lim_{T\to\infty}\frac{1}{T}\int_0^T u(x_k,t)dt$$

Substitute and average:

$$\cancel{\frac{\partial U_i}{\partial t}} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\frac{\mu}{\rho}\frac{\partial U_i}{\partial x_j}\right) + \frac{\partial\left(-\overline{u_i' u_j'}\right)}{\partial x_j}$$

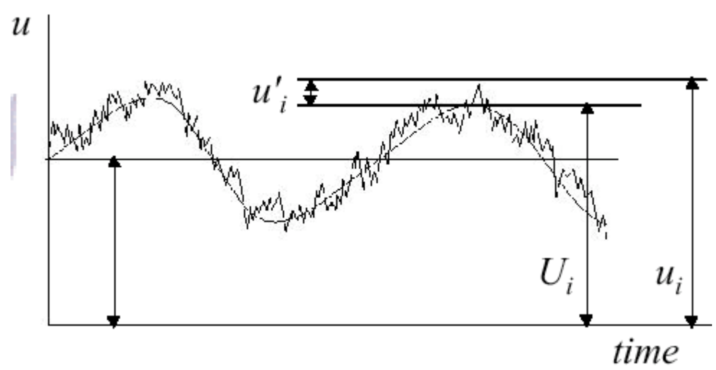
$$R_i j = -\overline{u_i' u_j'}$$

Closure problem

Unsteady RANS?

Every turbulent flow is unsteady BUT not all the unsteadiness is *turbulence*!

RANS averaging based on time average can be applied only to "statistically" steady flows. What if flow has a large scale periodicity (vortex shedding)?

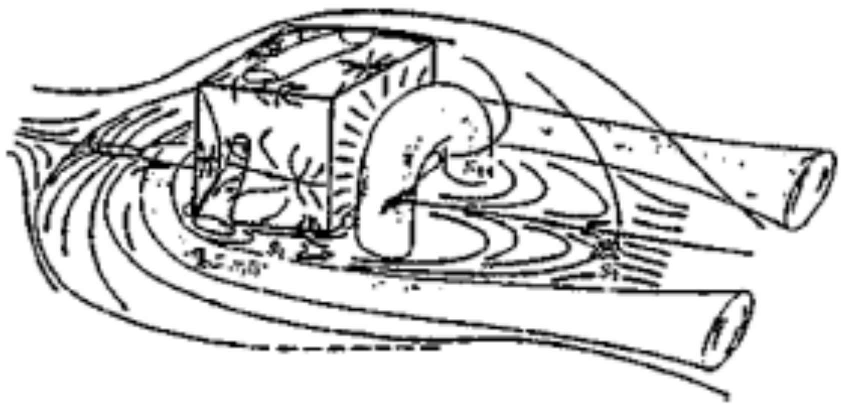


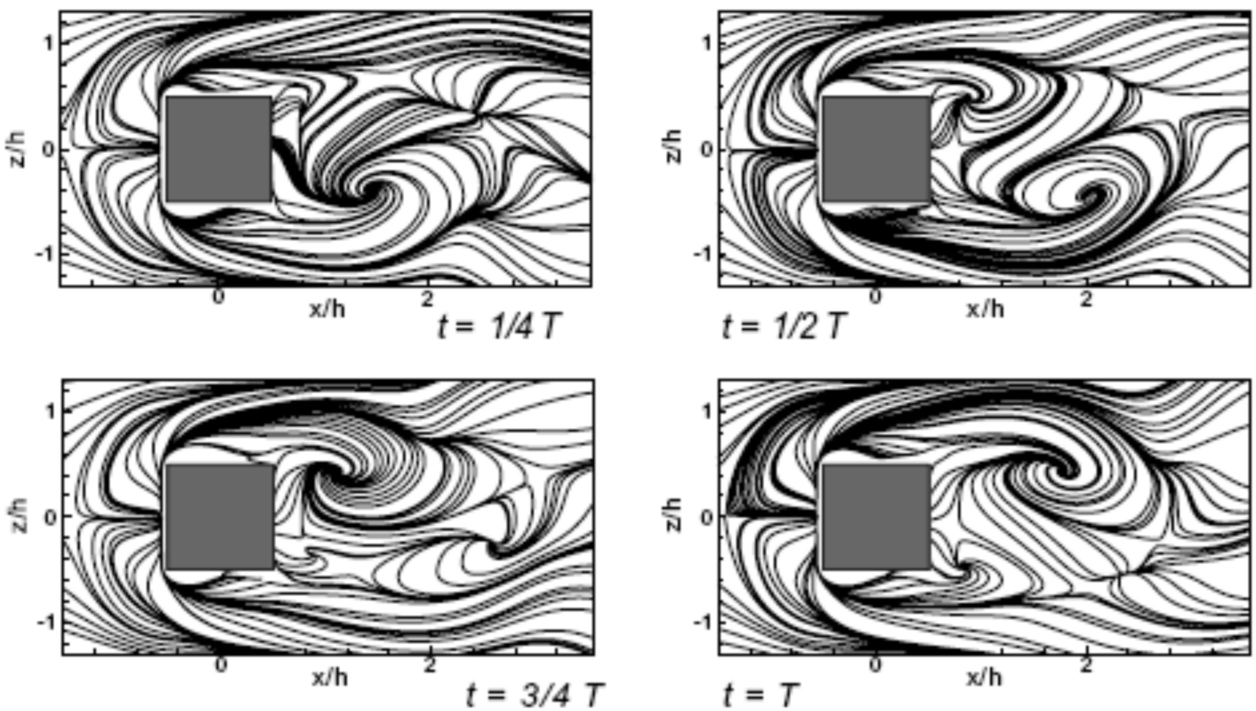
We can define the Reynolds-Averaging procedure in terms of Ensemble Average:

$$U_i(x_k,t) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_i^{(n)}(x_k,t)$$

$$u_i(x_k,t) = U_i(x_k,t) + u'(x_k,t)$$

Turbulent Vortex Shedding





Turbulent Vortex Shedding