Dual Interpretations and Applications

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https://web.stanford.edu/class/msande211x/handout.shtml

Guest Lecture of

ENGG 5501: Foundations of Optimization, CUHK

Chapters 2-3, LY 5thEdition

The LP Primal and Dual Pair

- Every optimization problem is associated with another optimization problem called dual (the original problem is called primal).
- Every variable of the dual is the "Lagrange multiplier" associated with a constraint in the primal, while every constraint of the dual is associated with a variable in the primal
- The dual is max (min) if the primal is min (max)
- The two optimal objective values are lower or upper bound for each other and they are equal if both are feasible.
- The optimal solution of the dual is the optimal "Lagrange multiplier" or shadow price vector of the primal, and vice versa

General Way to Construct the LP Dual

obj. coef. Vector	right-hand-side
right-hand-side	obj. coef. vector
A	A^T
Max model	Min model
$x_j \geq 0$	jth constraint $\geq$
$x_j \leq 0$	jth constraint $\leq$
x_j free	jth constraint $=$
ith constraint $\leq$	$y_i \geq 0$
ith constraint $\geq$	$y_i \leq 0$
ith constraint $=$	y_i free

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \mathbf{b}, \quad (\boldsymbol{y}) \\ & \mathbf{x} \geq \boldsymbol{0}. \end{aligned}$$

$$\begin{aligned} \max \quad & \boldsymbol{b}^T\boldsymbol{y} \\ \text{s.t.} \quad & A^T\boldsymbol{y} \leq \boldsymbol{c}, \end{aligned}$$

The dual of the dual is the primal: either side can be the primal

Possible Combination of Primal and Dual

Dual \ Primal	F-B	F-UB	IF
F-B	☺		
F-UB			☹
IF		☹	☹

$$
\begin{array}{llcll}
\min & -x_1 & - & x_2 & \\
\text{s.t.} & x_1 & - & x_2 & = 1 \\
 & -x_1 & + & x_2 & = 1 \\
 & x_1, & & x_2 & \geq 0
\end{array}
$$

$$
\begin{array}{llcll}
\max & y_1 & + & y_2 & \\
\text{s.t.} & y_1 & - & y_2 & \leq -1 \\
 & -y_1 & + & y_2 & \leq -1
\end{array}
$$

Two-Person Zero-Sum Matrix Game

$$\begin{pmatrix} 3 & -1 & -3 \\ -3 & 1 & 4 \end{pmatrix} = P$$

P is the payoff matrix of a two-person, ”Column” and ”Row”, zero-sum game.
Player Column Player chooses column(s) to maximize the payoff to Column
Player Row chooses row(s) to minimize the payoff to Column

Pure Strategy: Each player chooses a single column (row).

Mixed or Randomized Strategy: Each player randomly chooses columns (rows) strategies with a fixed probability distribution.

Nash Equilibrium: No player can alter its probability distribution to achieve better expected payoff.

Two-Person Zero-Sum Matrix Game II

$$\begin{pmatrix} 3 & -1 & -3 \\ -3 & 1 & 4 \end{pmatrix}$$

Player Column Player: probabilities x_1 to choose column 1, x_2 to choose column 2, and x_3 to choose column 3. Then the expected payoff is

$3x_1 - x_2 - 3x_3$ if Player Row chooses row 1

$-3x_1 + x_2 + 4x_3$ if Player Row chooses row 2

Thus, Player Column would

$$\text{maximize}_{(x1,x2,x3)} \quad \min\{3x_1 - x_2 - 3x_3,\ -3x_1 + x_2 + 4x_3\}$$
$$\text{s.t.} \quad X_1 + x_2 + x_3 = 1,\ (x_1, x_2, x_3) \geq 0$$

which can be cast as a linear program

$$\begin{array}{llll} \text{maximize}_{(x1,x2,x3,v)} & v & & \\ \text{s.t.} & -3x_1 + x_2 + 3x_3 + v \leq 0 & & y_1 \\ & 3x_1 - x_2 - 4x_3 + v \leq 0 & & y_2 \\ & x_1 + x_2 + x_3 = 1, & & u \\ & (x_1, x_2, x_3) \geq 0 & & \end{array}$$

Two-Person Zero-Sum Matrix Game III

$$\begin{pmatrix} 3 & -1 & -3 \\ -3 & 1 & 4 \end{pmatrix}$$

Then, the dual of the linear program

$$\begin{aligned}
\text{minimize}_{(y1,y2,u)} \quad & u \\
\text{s.t.} \quad & u - (3y_1 - 3y_2) \geq 0 \\
& u - (-y_1 + y_2) \geq 0 \\
& u - (-3y_1 + 4y_2) \geq 0 \\
& y_1 + y_2 = 1, \ (y_1, y_2) \geq 0
\end{aligned}$$

Interpretations:

Player Row: probabilities y_1 to choose row 1, y_2 to choose row 2. Then the expected payoff to Player Column is

$3y_1 - 3y_2$ if Player Column chooses column 1

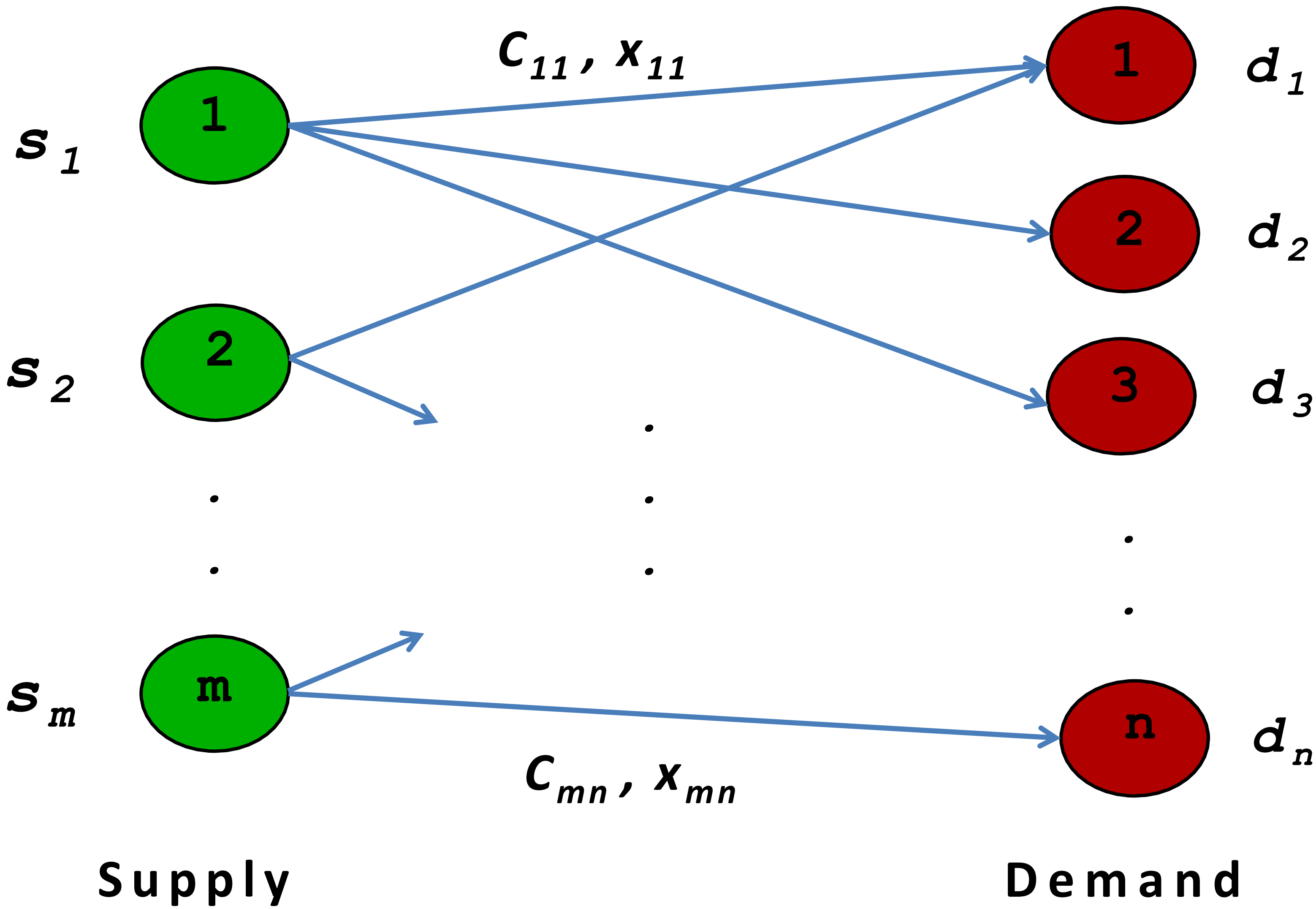
$-y_1 + y_2$ if Player Column chooses column 2

$-3y_1+4y_2$ if Player Column chooses column 3;

and Player Row does

$$\begin{aligned}
\text{minimize}_{(y1,y2)} \quad & \max\{3y_1 - 3y_2, -y_1 + y_2, -3y_1+4y_2\} \\
\text{s.t.} \quad & y_1+y_2 = 1, \ (y_1,y_2) \geq 0
\end{aligned}$$

The Transportation Problem and its Dual



The Transportation Dual Interpretation

Primal

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = s_i, && \forall\, i = 1,\ldots,m \\
& \sum_{i=1}^{m} x_{ij} = d_j, && \forall\, j = 1,\ldots,n \\
& x_{ij} \geq 0, && \forall\, i, j
\end{aligned}
$$

Dual

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \\
\text{s.t.} \quad & u_i + v_j \leq c_{ij}, && \forall\, i, j
\end{aligned}
$$

Shipping Company's new charge scheme:

u_i: supply site unit charge

v_i: demand site unit charge

$u_i + v_j \leq c_{ij}$: competitiveness

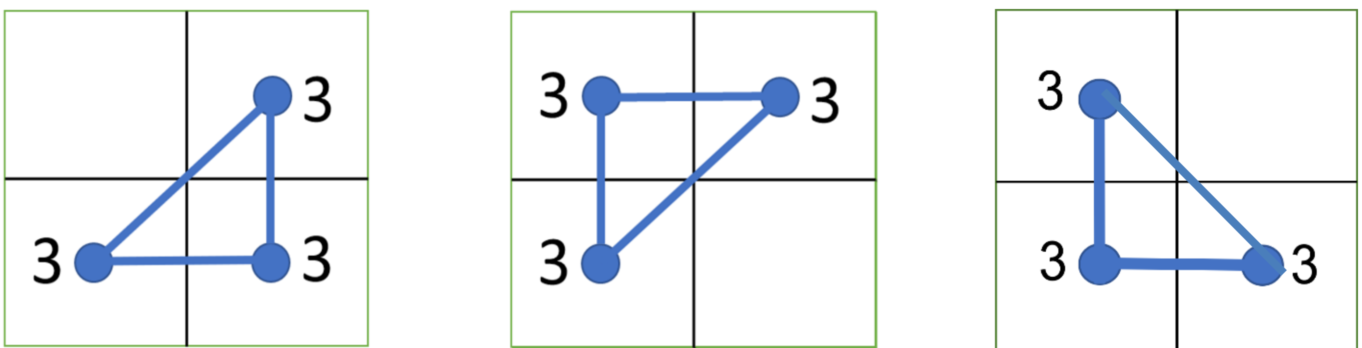
Machine Learning: The Wassestein Barycenter Problem I

The minimal transportation cost in Data Science is called the Wasserstein distance between a supply distribution and a demand distribution.
The **Wasserstein Barycenter Problem** is to find a distribution/points such that the sum of its Wasserstein distances to each of a set of distributions/points would be minimized

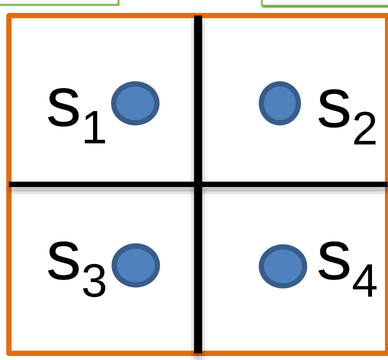
$$\min_s \sum_k \mathbf{WD(s, d^k)} \textbf{ s.t. total mass constraint}$$

WD(s, d^k)=

$$\begin{aligned}
\min \quad & \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} = s_i, \quad \forall\ i = 1, \dots, N \\
& \sum_{i=1}^{N} x_{ij} = d_j, \quad \forall\ j = 1, \dots, N \\
& x_{ij} \geq 0, \qquad \forall\ i, j
\end{aligned}$$



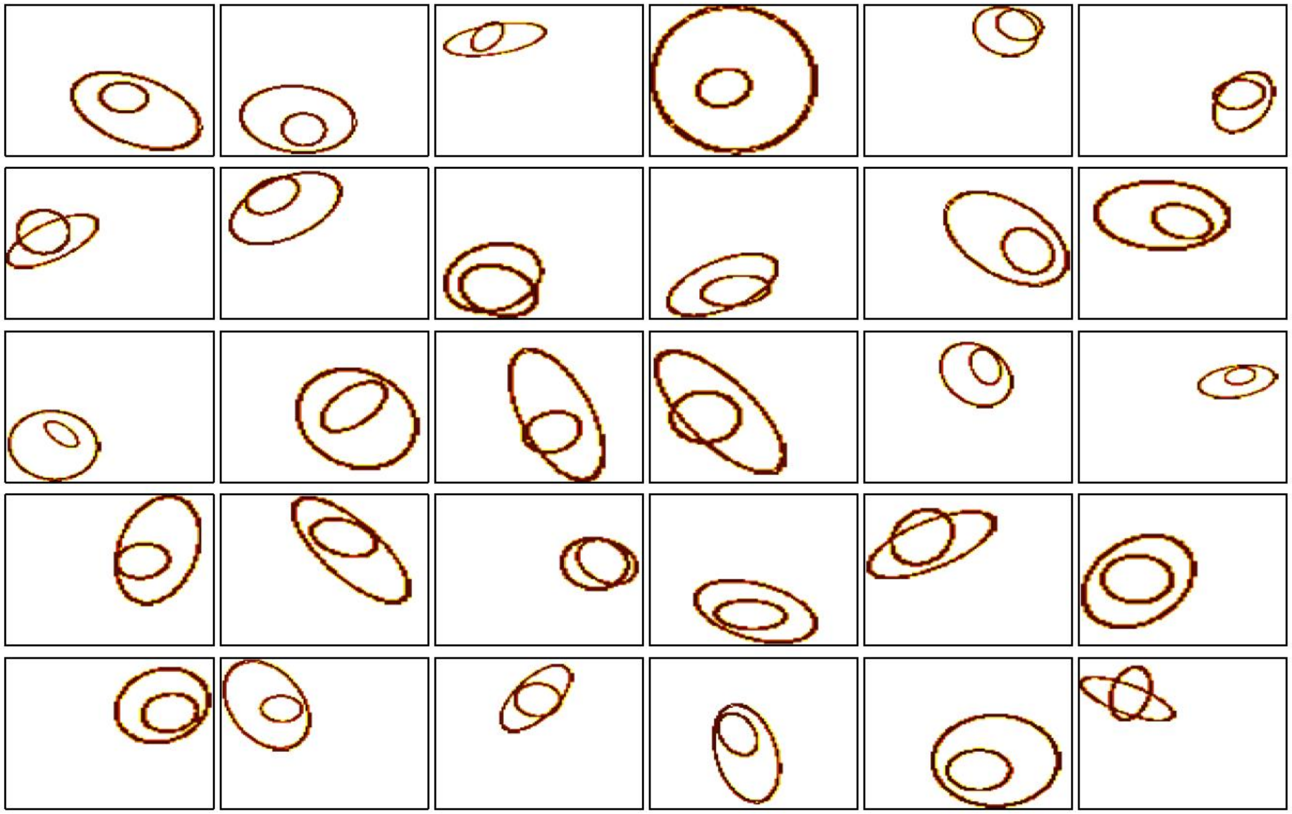
Three possible demand distribution scenario of 4 cities



Constraints:
$s_1+s_2+s_3+s_4=9$
$(s_1,s_2,s_3,s_4)>=0$

$$C=\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

Machine Learning: The Wassestein Barycenter Problem II



What is the best ``mean or consensus‘’ image from a set of images (pixel distributions)?

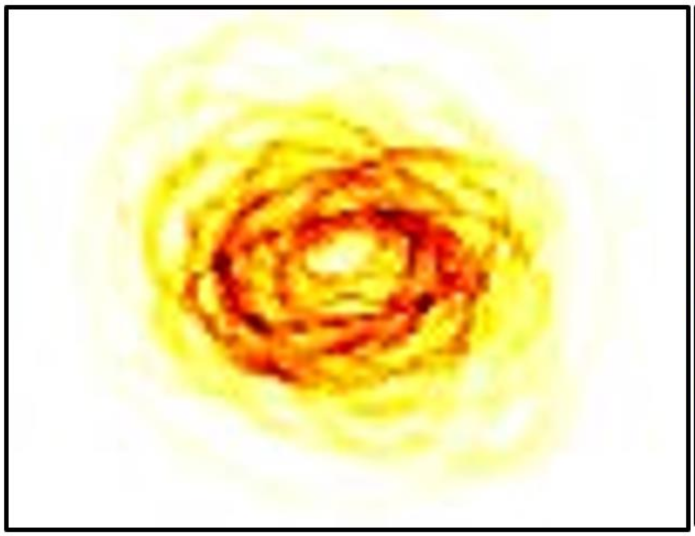
- Simple average
- Simple average after re-centering
- The Wasserstein Barycenter of the set of images (self re-center and rotation)

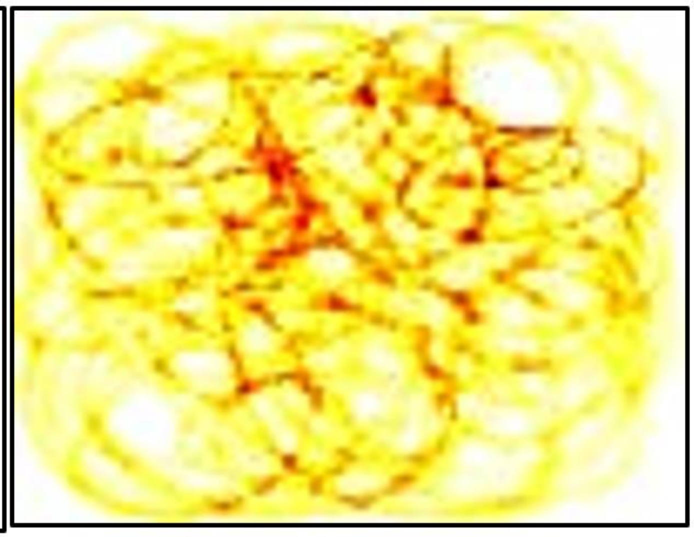
Machine Learning: The Wassestein Barycenter Problem III

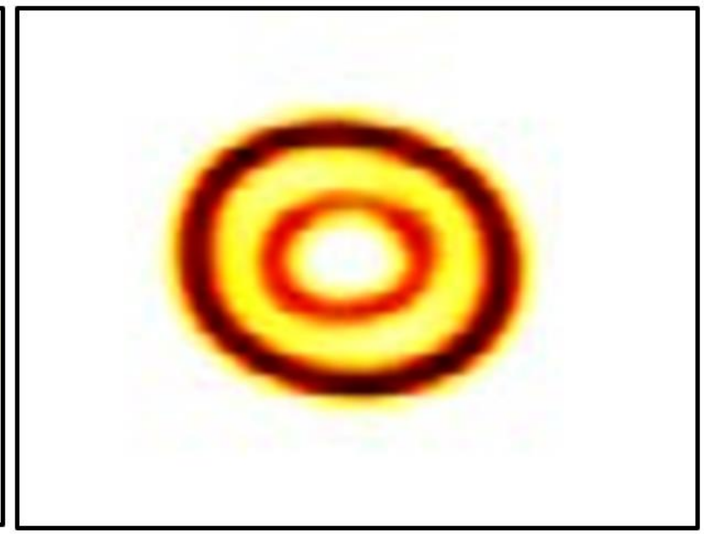
The simple avarage of n points is

$$\mathbf{s}=(\sum_k \mathbf{d}^k)/n \quad \text{or} \quad \min_{\mathbf{s}} \sum_{\mathbf{k}} (\|\mathbf{s}-\mathbf{d}^k\|_2)^2$$

The **Wasserstein Barycenter Problem** is to find a distribution/points such that the sum of its Wasserstein distances to each of a set of distributions/points would be minimized (self re-center and rotation).







Simple average after re-centering   Simple average   the Barycenter image

Interpretation of Dual: Sensitivity Analyses of *b*

$$
\boxed{\begin{aligned} OV(b) := \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0 \end{aligned}}
$$

$$
\begin{array}{llllllll}
\min & -x_1 & -2x_2 & & & & \\
\text{s.t.} & x_1 & & +x_3 & & & = 1 \\
 & & x_2 & & +x_4 & & = 1 \\
 & x_1 & +x_2 & & & +x_5 & = 1.5 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0.
\end{array}
$$

$OV(\boldsymbol{b})$ is a convex function of $\boldsymbol{b}$ and $\nabla OV(\boldsymbol{b}) = \boldsymbol{y}^*$

$\boldsymbol{y} = (0, \ -1, \ -1)^T$

If b1 is increased or decreased a little, does OV change?

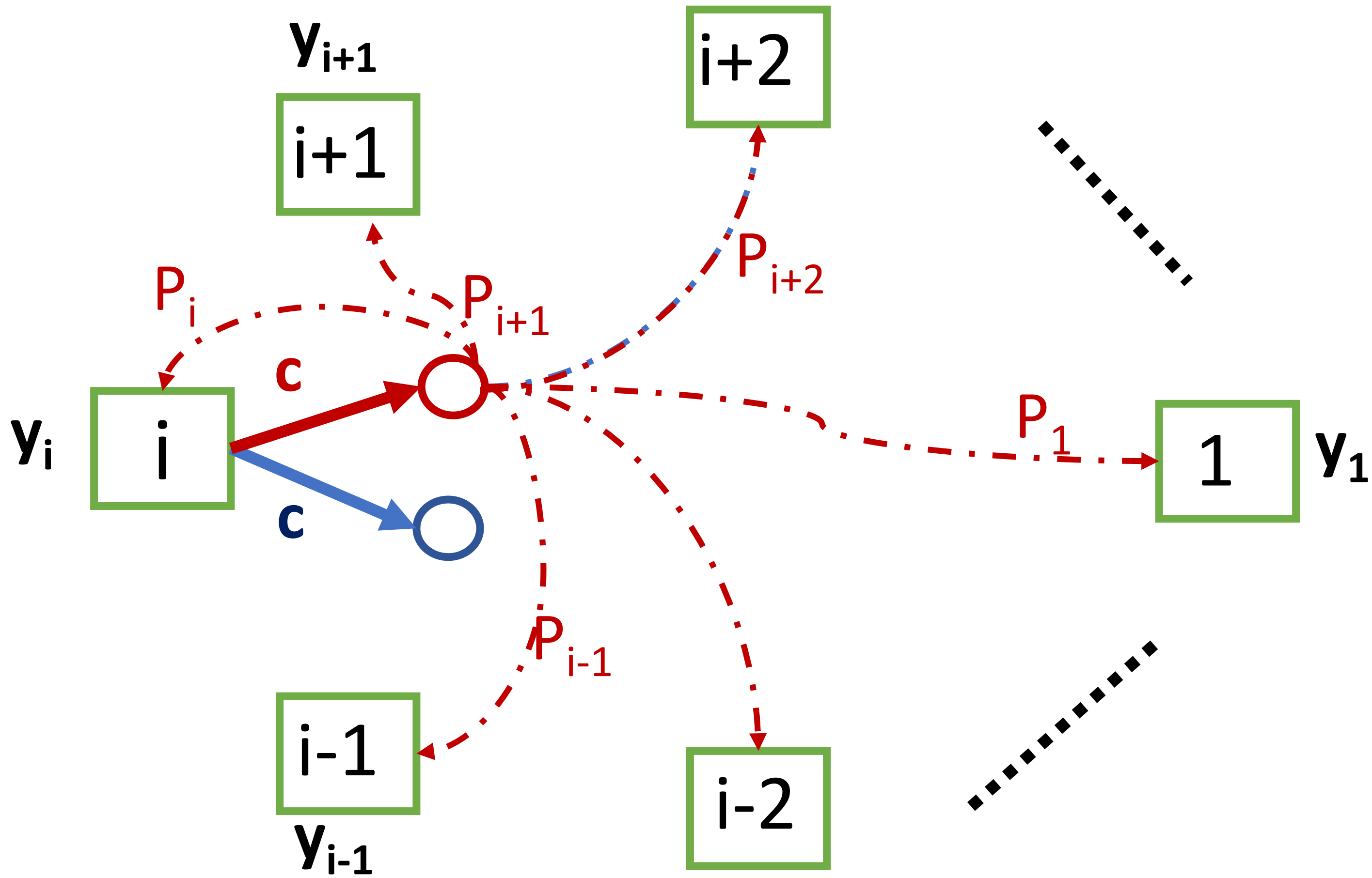
If b2 is increased or decreased a little, does OV change? How much?

If b3 is increased or decreased a little, does OV change? How much?

Reinforcement Learning and Markov Decision Process

- Markov decision process provides a mathematical framework for modeling sequential decision-making in situations where outcomes are partly random and partly under the control of a decision maker, and it is called Reinforcement Learning lately.
- MDPs are useful for studying a wide range of optimization problems solved via dynamic programming, where it was known at least as early as the 1950s (cf. Shapley 1953, Bellman 1957).
- Modern applications include dynamic planning, social networking, and almost all other dynamic/sequential-decision-making problems in Mathematical, Physical, Management, Economics, and Social Sciences.
- MDP is characterized by States and Actions; and at each time step, the process is in a state and the decision maker chooses an action to optimize a long-term goal.

MDP/RL State/Action Environment



$$c \quad + \quad \gamma \quad p^T y$$

immediate cost expect future cost

Cost-to-Go values and the LP formulation

- In general, let $y \in R^m$ represent the expected present cost-to-go values of the *m* states, respectively, for a given policy. Then, the cost-to-go vector of the optimal policy, with the discount factor γ, by Bellman's Principle is a Fixed Point:

$$y_i = \min\{ c_j + \gamma p_j^T y, j \in A_i\}, \forall i,$$

$$j_i = \arg\min\{ c_j + \gamma p_j^T y, j \in A_i\}, \forall i.$$

- Such a fixed-point computation can be formulated as an LP

$$\begin{aligned} \max \quad & \sum_i y_i \\ \text{s.t.} \quad & y_i \le c_j + \gamma p_j^T y, \forall j \in A_i; \forall i. \end{aligned}$$

- The maximization is trying to pushing up each yi to the highest possible so that it equal to min-argument. When the optimal y is found, one can then find the index of the original optimal action/policy using argmin.

Consider a Simplified MDP-RL Problem (Maze-Run)

$$\max\ y_0 + y_1 + y_2 + y_3 + y_4 + y_5$$

$$\text{s.t.}\quad y_5 \le 0 + \gamma y_5$$

$$y_4 \le 1 + \gamma y_5$$

$$y_3 \le 0 + \gamma y_4$$

$$y_3 \le 0 + \gamma y_5$$

$$y_2 \le 0 + \gamma y_3$$

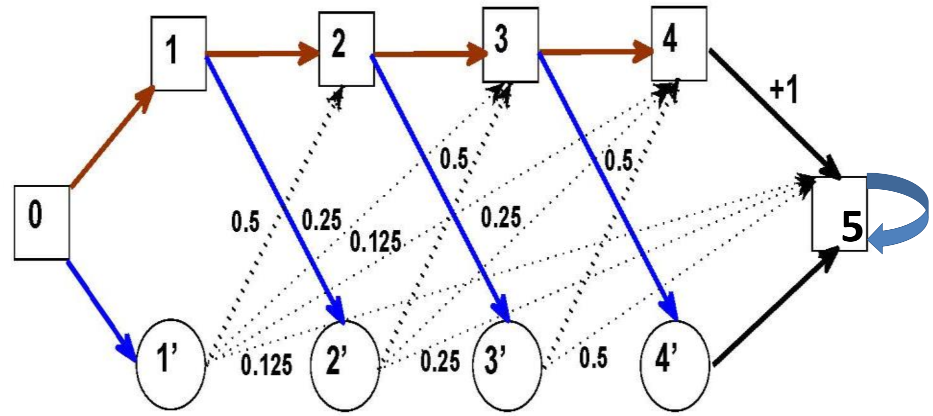
$$y_2 \le 0 + \gamma(0.5 y_4 + 0.5 y_5)$$

$$y_1 \le 0 + \gamma y_2$$

$$y_1 \le 0 + \gamma(0.5 y_3 + 0.25 y_4 + 0.25 y_5)$$

$$y_0 \le 0 + \gamma y_1$$

$$y_0 \le 0 + \gamma(0.5 y_2 + 0.25 y_3 + 0.125 y_4 + 0.125 y_5)$$



- y_i: expected overall cost if stating from State i.
- State 4 is a trap
- State 5 is the destination
- Each other state has two options: Go directly to the next state OR a short-cut go to other states with uncertainties

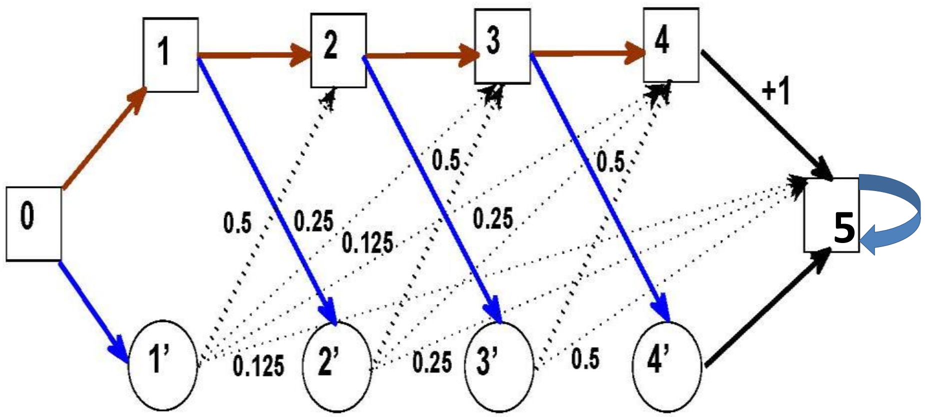
$\mathbf{y^*_0=y^*_1=y^*_2=y^*_3=y^*_5=0}$

$\mathbf{y^*_4=1}$

Physical Interpretation of the Maze-Run Dual

$$\max\ y_0 + y_1 + y_2 + y_3 + y_4 + y_5$$

$$
\begin{aligned}
\text{s.t.}\quad & y_5 \le 0 + \gamma y_5 \quad (x_5) \\
& y_4 \le 1 + \gamma y_5 \quad (x_4) \\
& y_3 \le 0 + \gamma y_4 \quad (x_{3r}) \\
& y_3 \le 0 + \gamma y_5 \quad (x_{3b}) \\
& y_2 \le 0 + \gamma y_3 \quad (x_{2r}) \\
& y_2 \le 0 + \gamma(0.5y_4 + 0.5y_5) \quad (x_{2b}) \\
& y_1 \le 0 + \gamma y_2 \quad (x_{1r}) \\
& y_1 \le 0 + \gamma(0.5y_3 + 0.25y_4 + 0.25y_5) \quad (x_{1b}) \\
& y_0 \le 0 + \gamma y_1 \quad (x_{0r}) \\
& y_0 \le 0 + \gamma(0.5y_2 + 0.25y_3 + 0.125y_4 + 0.125y_5) \quad (x_{0b})
\end{aligned}
$$



x_j represents (discounted) how many expected times (frequency) actions j being taken in a policy.

The Dual of the Maze Example

x:	$(0r)$	$(0b)$	$(1r)$	$(1b)$	$(2r)$	$(2b)$	$(3r)$	$(3b)$	(4)	(5)	b
c:	0	0	0	0	0	0	0	0	1	0	
(0)	1	1	0	0	0	0	0	0	0	0	1
(1)	$-\gamma$	0	1	1	0	0	0	0	0	0	1
(2)	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	0	0	1
(3)	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	1
(4)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	0	1
(5)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	0	$-\gamma$	$-\gamma$	$1-\gamma$	1

min $\boldsymbol{c}^T\boldsymbol{x}$

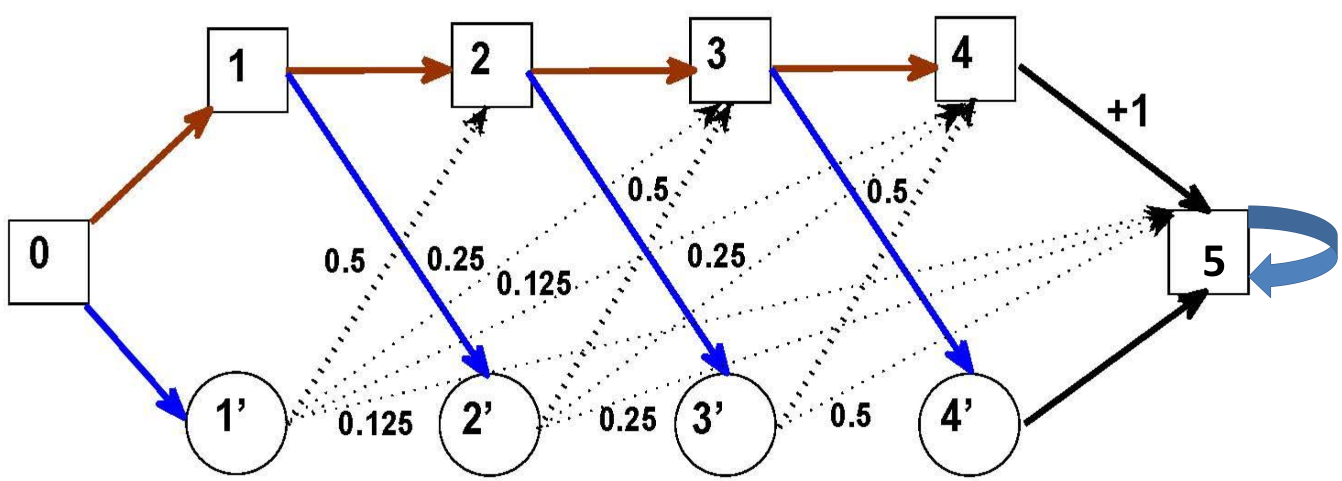
s.t. $A\boldsymbol{x} = \mathbf{e}, \ (\boldsymbol{y})$

$\boldsymbol{x} \geq \boldsymbol{0}.$

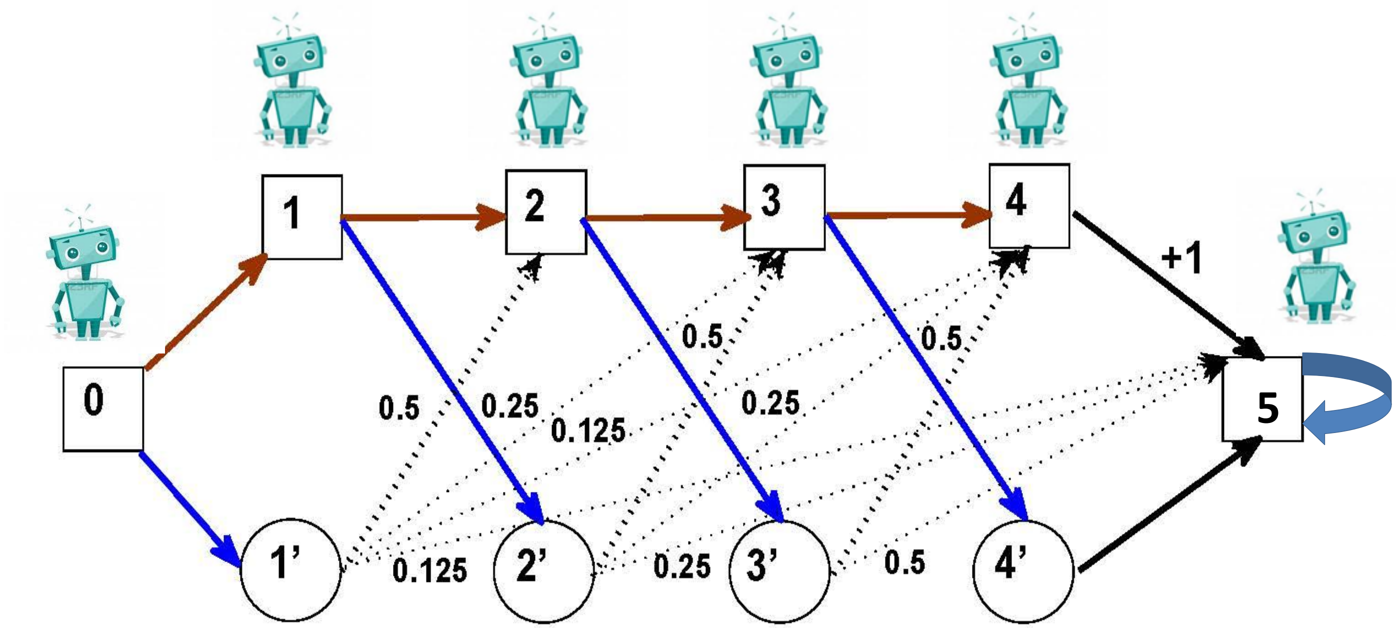
The optimal dual solution is

$$x^*_{0r} = 1,\ x^*_{1r} = 1+\gamma,\ x^*_{2r} = 1+\gamma+\gamma^2,\ x^*_{3b} = 1+\gamma+\gamma^2+\gamma^3,\ x^*_4 = 1,$$

$$x^*_5 = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma}.$$

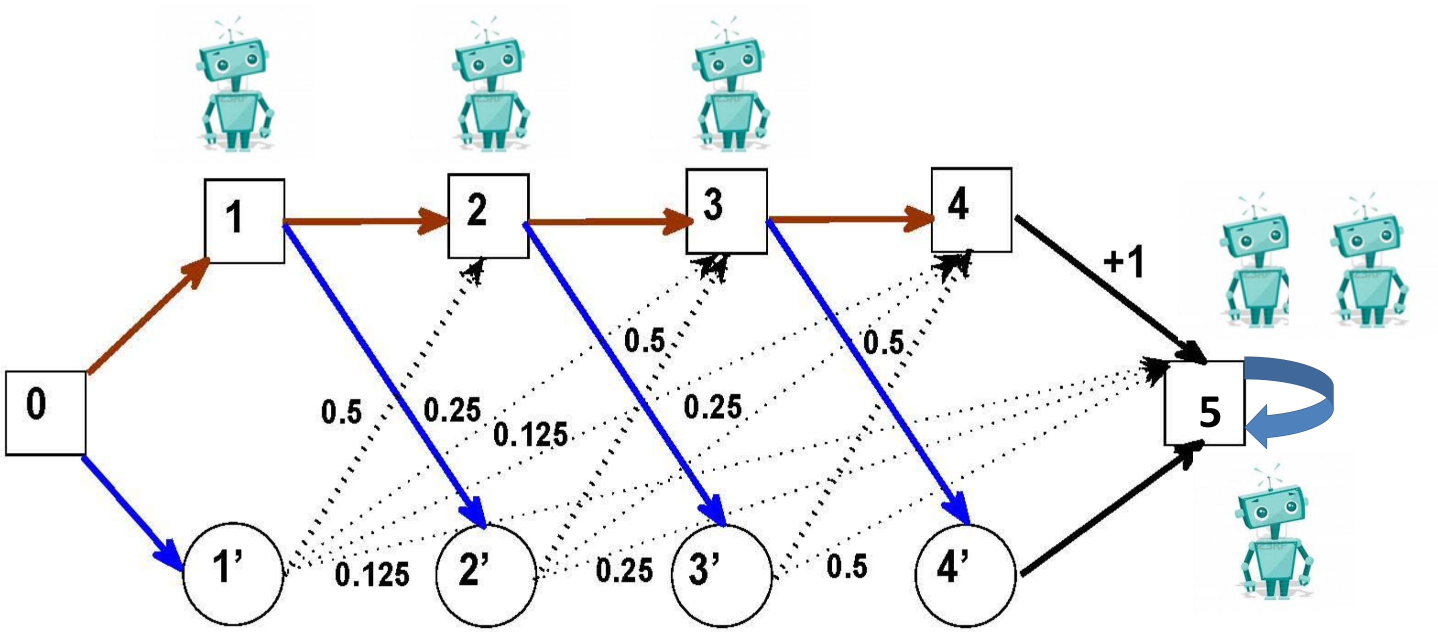


Time 1



0
1
2
3
4
5
1'
2'
3'
4'
+1
0.5
0.25
0.125
0.5
0.25
0.5
0.125
0.25
0.5

Time 2



0
1
2
3
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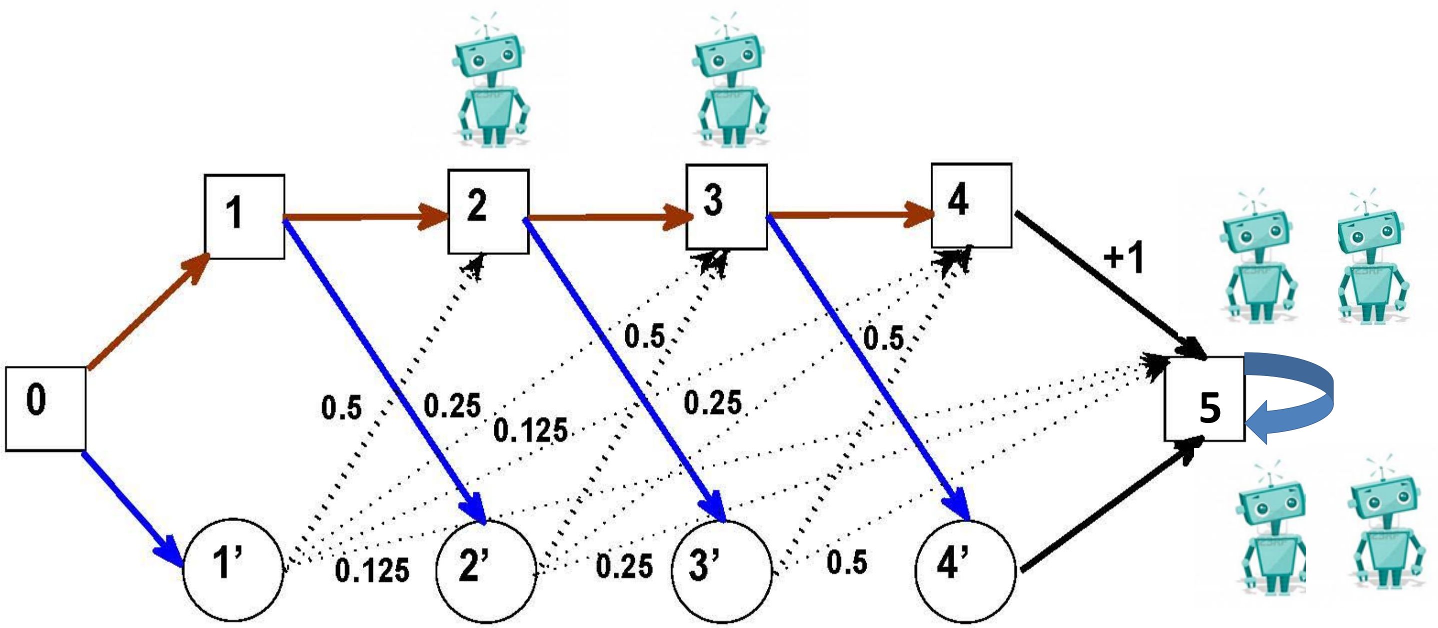
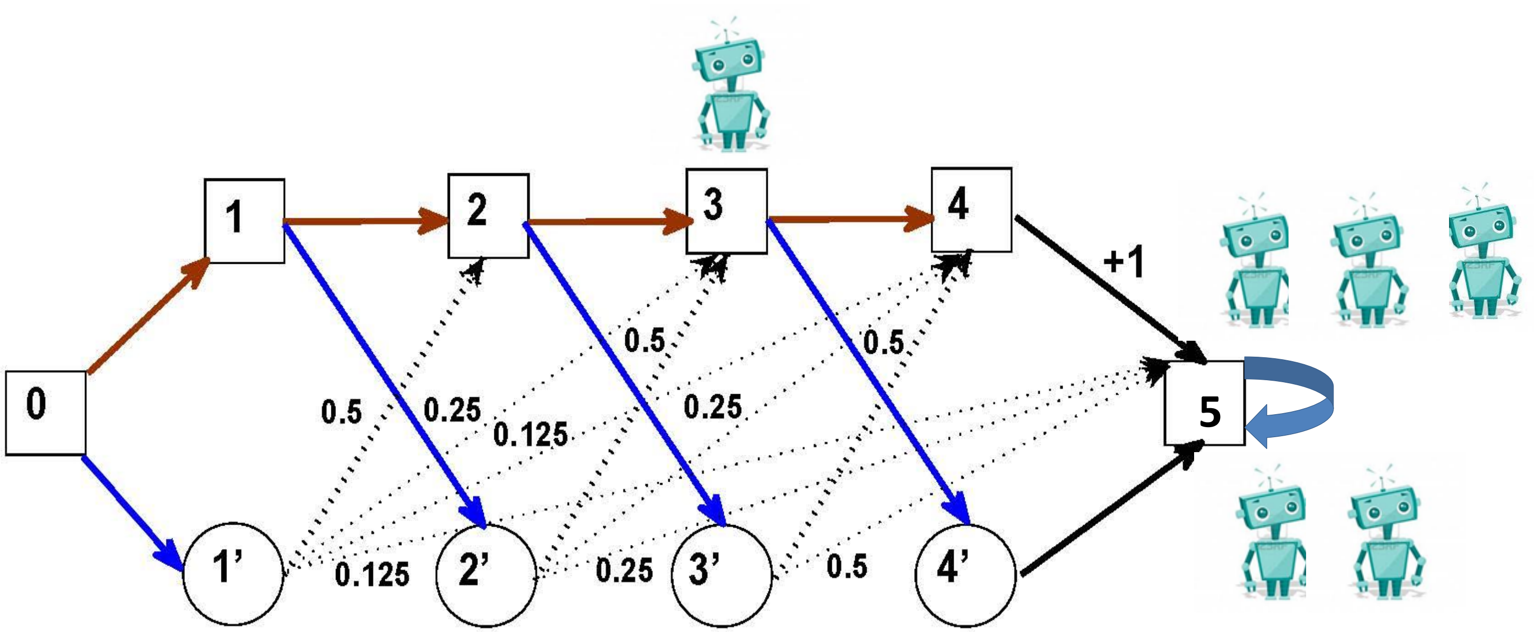
1'
2'
3'
4'
+1
0.5
0.25
0.125
0.5
0.25
0.5
0.125
0.25
0.5

Time 3



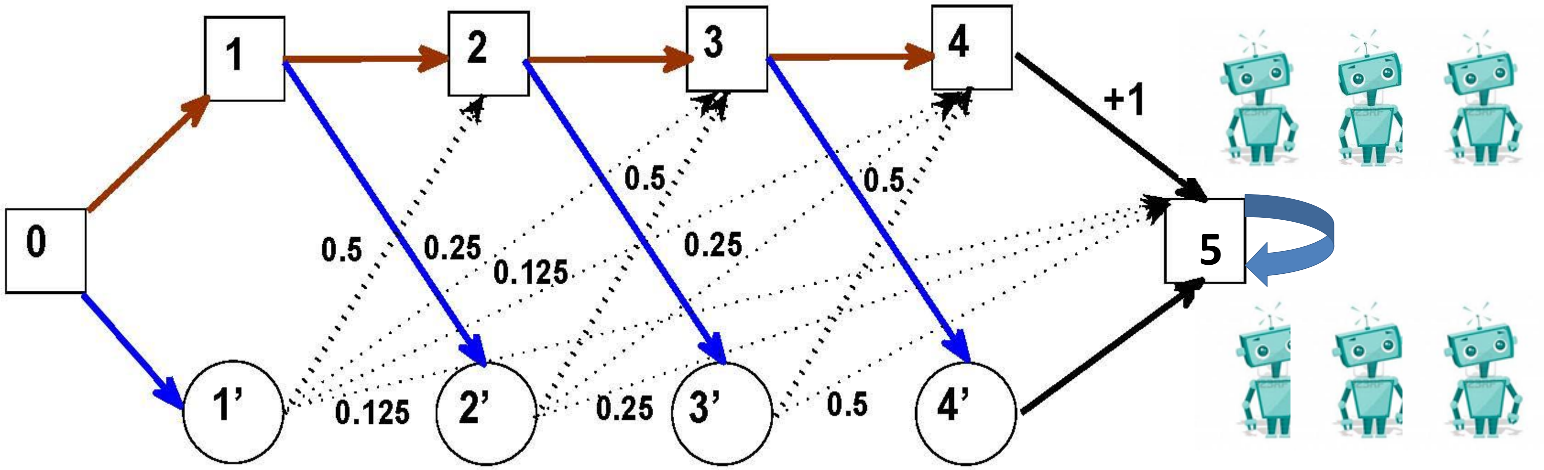
0
1
2
3
4
5
1'
2'
3'
4'
+1
0.5
0.25
0.125
0.5
0.25
0.5
0.125
0.25
0.5

Time 4



0
1
2
3
4
5
1'
2'
3'
4'
+1
0.5
0.25
0.125
0.5
0.25
0.5
0.125
0.25
0.5

Time 5



Recall the optimal dual solution values are:

$$x_{0r}^* = 1,\ x_{1r}^* = 1+\gamma,\ x_{2r}^* = 1+\gamma+\gamma^2,\ x_{3b}^* = 1+\gamma+\gamma^2+\gamma^3,\ x_4^* = 1,$$

$$x_5^* = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma}.$$

Information/Prediction Market
(Peters, So and Y 2007)

- A place where information is aggregated via market for the primary purpose of forecasting events.

- **Why:**
  - Wisdom of the Crowds: Under the right conditions groups can be remarkably intelligent and possibly smarter than the smartest person.
    James Surowiecki
  - Efficient Market Hypothesis: financial markets are “informationally efficient”, prices reflect all known information
- **Market for Betting the World Cup Winner**
  - Assume 5 teams have a chance to win the World Cup: Argentina, Brazil, Italy, Germany and France

A Market Platform

- **Market for World Cup Winner**
  - We'd like to have a standard payout of $1 per share if a participant has a winning order.
- **List of Combinatorial Orders**

Order	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
1	0.75	10	1	1	1		
2	0.35	5				1	
3	0.40	10	1		1		1
4	0.95	10	1	1	1	1	
5	0.75	5		1		1	

Market maker: Order fill - how many shares to sell for each order?

The Abstract Market Platform

- Given *m* states that are mutually exclusive and exactly one of them will be realized at the maturity.
- An order is a bet on one or a combination of states
  - $(a_{i1}, a_{i2}, \ldots, a_{im})$: the entry value is 1 if the jth state is included in the winning basket and 0 other wise.
- with a price limit
  - π_i : the maximum price the participant is willing to pay for one share of the order
- and a share quantity limit
  - q_i : the maximum number of shares the participant is willing to buy.
- A contract agreement so that on maturity it is worth a notional one dollar per share if the order includes the winning state and worth 0 otherwise.

Market-Maker's Decision Problem I

- Let x_i be the number of shares sell to order i.
- The revenue collected for the sale:

$$\sum_i \pi_i x_i$$

$$0.75x_1 + \ldots + 0.75x_5$$

Order fill	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
x1	0.75	10	1	1	1		
x2	0.35	5				1	
x3	0.40	10	1		1		1
x4	0.95	10	1	1	1	1	
x5	0.75	5		1		1	

- The cost depends on which team wins:
  - If jth team wins (for example, if Brazil wins in the example):

$$\sum_i a_{ij} x_i$$

$$x_1 + x_4 + x_5$$

- We consider the worse case cost and profit

$$\max_{j=1,\ldots,m}\{\sum_i a_{ij} x_i\} \Rightarrow \max \ \left(\sum_i \pi_i x_i - \max_{j=1,\ldots,m}\{\sum_i a_{ij} x_i\}\right)$$

Market-Maker's Decision Problem II

$$\begin{aligned} \max \quad & \sum_i \pi_i x_i - \max_j \{ \sum_i a_{ij} x_i \} \\ \text{s.t.} \quad & 0 \le x_i \le q_i \qquad \forall\, i = 1, \dots, n \end{aligned}$$

Collected revenue

Worst-case cost

Cost if state j is realized

$$\begin{aligned} \max \quad & \sum_i \pi_i x_i - w \\ \text{s.t.} \quad & \sum_i a_{ij} x_i \le w \qquad \forall\, j \in S \\ & 0 \le x_i \le q_i \qquad \forall\, i \in N \end{aligned}$$

This is an LP problem; later you will learn that the optimal dual solution gives prices of each team

Compact Coefficients

Order	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
1	0.75	10	1	1	1		
2	0.35	5				1	
3	0.40	10	1		1		1
4	0.95	10	1	1	1	1	
5	0.75	5		1		1	

π q A

The Dual of the LP Problem

Corresponding Dual Variables

$$
\begin{array}{lll}
\max & \pi^T \mathbf{x} - x_{n+1} & \\
\text{s.t.} & A^T \mathbf{x} - \mathbf{1} \cdot x_{n+1} \le \mathbf{0} & \boxed{\boldsymbol{p}} \\
& \mathbf{x} \le \mathbf{q} & \boxed{\boldsymbol{s}} \\
& \mathbf{x} \ge \mathbf{0} & \\
& x_{n+1} \text{ free} &
\end{array}
$$

where $\mathbf{1}$ is the vector of all ones.

$\pi^T \mathbf{x}$: the revenue amount can be collected.

x_{n+1}: the worst-case cost (amount need to pay to the winners).

The Dual: Regression with "Under-Bid" Eliminating

$$
\begin{array}{lll}
\min & \boldsymbol{q}^T \boldsymbol{s} & \\
\text{s.t.} & A\boldsymbol{p} + \boldsymbol{s} & \geq \boldsymbol{\pi}, \\
& -\boldsymbol{1}^T \boldsymbol{p} & = -1, \\
& (\boldsymbol{p}, \boldsymbol{s}) & \geq 0.
\end{array}
$$

$\boldsymbol{p}_j$: the shadow/dual price of state j;

$\boldsymbol{a}_i\boldsymbol{p}$: the ith order unit cost at prices $\boldsymbol{p}$;

$\boldsymbol{s}_j$: the unit profit from the jth order ($\boldsymbol{s}=\max\{\boldsymbol{0}, \boldsymbol{\pi}-A\boldsymbol{p}\}$)

The dual problem is to minimize the total "Regression Loss" collected from the (competitive or high-bid) orders, $\boldsymbol{q}^T\boldsymbol{s}$.

ReLu-Regression for Probability Distribution/Information

$$\begin{aligned} \min \quad & \boldsymbol{q}^T \max\{\boldsymbol{0}, \boldsymbol{\pi}\text{-}A\boldsymbol{p}\} \\ \text{s.t.} \quad & \boldsymbol{1}^T \boldsymbol{p} = 1, \\ & \boldsymbol{p} \geq 0 \end{aligned}$$

$\boldsymbol{p}_j$: the shadow-price/probability estimation of state j;

$\boldsymbol{a}_i\boldsymbol{p}$: the ith order unit cost at prices $\boldsymbol{p}$;

$\boldsymbol{\pi}_i$: the ith order bidding price;

$\boldsymbol{q}_i$: the ith order quantity limit;

The dual problem is to minimize the total weighted discrepancy among the competitive bidders such that all winners' betting beliefs $\boldsymbol{\pi}$ are fully utilized, while under-bidders (outliers) would be automatically removed from the estimation.

The World Cup Betting Example

Orders Filled

Order	Price Limit	Quantity Limit	Filled	Argentina	Brazil	Italy	Germany	France
1	0.75	10	5	1	1	1		
2	0.35	5	5				1	
3	0.40	10	5	1		1		1
4	0.95	10	0	1	1	1	1	
5	0.75	5	5		1		1	

State Prices

	Argentina	Brazil	Italy	Germany	France
Price	0.20	0.35	0.20	0.25	0.00

Duality Application in Financial Markets

- Create a market for World Cup security exchange
  - Five securities in a market for open trading at fixed prices and pay-offs, and short is allowed
  - We'd like to decide how many shares to purchase or sell to maximize the worst case pay-off when the game is realized.

Security	Price π	Share Limit q	Argentina	Brazil	Italy	Germany	France
1	\$0.75	∞	\$1	\$1	\$1		
2	\$0.35	∞				\$1	
3	\$0.40	∞	\$1		\$1		\$1
4	\$0.95	∞	\$1	\$1	\$1	\$1	
5	\$0.75	∞		\$1		\$1	

Portfolio Optimization Model

No share limit, and short is allowed:

$$
\begin{array}{ll}
\max & s - .75x_1 - .35x_2 - .4x_3 - .95x_4 - .75x_5 \\
\text{s.t.} & s - x_1 - x_3 - x_4 \le 0 \\
& s - x_1 - x_4 - x_5 \le 0 \\
& s - x_1 - x_3 - x_4 \le 0 \\
& s - x_2 - x_4 - x_5 \le 0 \\
& s \quad - x_3 \quad \le 0 \\
& \text{all variables free}
\end{array}
$$

Is this (primal) problem feasible?
Is this problem unbounded? – Check if the dual is feasible or not.

Dual of the Portfolio Optimization Model

$$
\begin{aligned}
\min \quad & 0^T \bullet p \\
\text{s.t.} \quad & p_1 + p_2 + p_3 + p_4 + p_5 = 1 \\
& p_1 + p_2 + p_3 = .75 \\
& p_4 = .35 \\
& p_1 + p_3 + p_5 = .4 \\
& p_1 + p_2 + p_3 + p_4 = .95 \\
& p_2 + p_4 = .75 \\
& p_1, p_2, p_3, p_4, p_5 \geq 0
\end{aligned}
$$

P: the security shadow prices

Is the dual feasible? Arbitrage exists if it is not!

On-Line Linear Programming

- Off-line Problem is an (0,1) linear program that can be relaxed as LP
- But now trader/Bidders come one by one sequentially,
- The retailer has to make the decision as soon as an order arrives with the arrived combinatorial order/bid $(\mathbf{a}_k, \pi_k)$
- The retailer faces a dilemma:
  - **To sell or not to sell – this is the decision**
- Optimal Policy or Online Algorithm?

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} \pi_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \forall i = 1, \ldots, m \\
& x_j = \{0 \; or \; 1\} \qquad \forall j = 1, \ldots, n
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} \pi_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \forall i = 1, \ldots, m \\
& 0 \le x_j \le 1 \qquad \forall j = 1, \ldots, n
\end{aligned}
$$

Off-Line LP Relaxation

CSC of Off-Line Retailer Linear Programming

- Let the optimal solution be $\boldsymbol{x}^*$ and the optimal shadow piece be $\boldsymbol{y}^*$
- Then from the CSC conditions:

  $x_j^* = 1$ if $\pi_j > \mathbf{a}_j^T \boldsymbol{y}^*$

  $x_j^* = 0$ if $\pi_j < \mathbf{a}_j^T \boldsymbol{y}^*$

  $x_j^* =$ fraction if $\pi_j = \mathbf{a}_j^T \boldsymbol{y}^*$

- If we know $\boldsymbol{y}^*$, the online decision would be easy!

$$
\begin{array}{lll}
\max & \sum_{j=1}^{n} \pi_j x_j & \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \forall i = 1, \ldots, m \\
 & 0 \leq x_j \leq 1 & \forall j = 1, \ldots, n
\end{array}
$$

Off-Line LP Relaxation

Online Algorithm and Price-Mechanism

- **Learn "ideal" itemized optimal prices**
- **Use the prices to price each bid**
- **Accept if it is a over bid, and reject otherwise**

Bid #	$100	$30	….	…	…	Inventory	Price?
Decision	x1	x2					
Pants	1	0	….	…	…	100	45
Shoes	1	0				50	45
T-Shirts	0	1				500	10
Jackets	0	0				200	55
Hats	1	1	…	…	…	1000	15

Such ideal prices exist, and they are shadow/dual prices of the offline LP

How to Learn the Shadow Prices Sequentially?

- Sequential Linear Programming Mechanism (SLPM)
  - Solving the LP based on immediately past several periods' data and use the resulted optimal shadow prices to make decision for the next period orders; and repeat when the current period is over.

- The shadow prices are updated periodically and being used to make online decisions for the next period.

Wait for Data from 1 to εn

- Set $x_j=0$ for $j=1,\ldots,\varepsilon n$.
- Solve LP:
- Let $\boldsymbol{p}^1$ be the optimal shadow price vector and use it to make online decision for orders from $\varepsilon n+1$ to $2\varepsilon n$.

$$
\begin{array}{lll}
\max & \sum_{j=1}^{\varepsilon n} \pi_j x_j & \\
\text{s.t.} & \sum_{j=1}^{\varepsilon n} a_{ij} x_j \le \varepsilon b_i & \forall i \\
 & 0 \le x_j \le 1 & \forall j
\end{array}
$$

Now Use All Data from 1 to 2εn

- Now solve LP:

$$
\begin{array}{lll}
\max & \displaystyle\sum_{j=1}^{2\varepsilon n} \pi_j x_j & \\
\text{s.t.} & \displaystyle\sum_{j=1}^{2\varepsilon n} a_{ij} x_j \le 2\varepsilon b_i & \forall i \\
 & 0 \le x_j \le 1 & \forall j
\end{array}
$$

- Let $\boldsymbol{p}^2$ be the optimal shadow price vector and use it to make online decision for orders from $2\varepsilon n+1$ to $4\varepsilon n$.

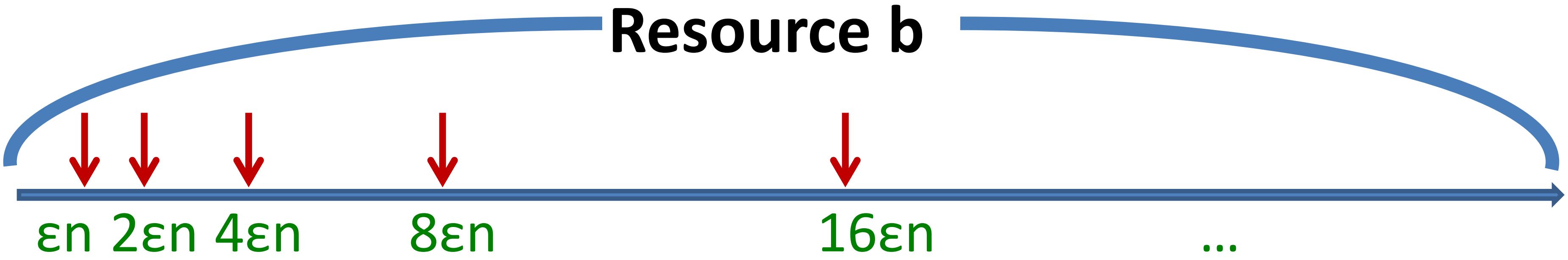
Now Use All Data from 1 to 4εn

- Now solve LP:

$$
\begin{array}{lll}
\max & \displaystyle\sum_{j=1}^{4\varepsilon n} \pi_j x_j & \\
\text{s.t.} & \displaystyle\sum_{j=1}^{4\varepsilon n} a_{ij} x_j \leq 4\varepsilon b_i & \forall i \\
& 0 \leq x_j \leq 1 & \forall j
\end{array}
$$

- Let $\boldsymbol{p}^3$ be the optimal shadow price vector and use it to make online decision for orders from $4\varepsilon n+1$ to $8\varepsilon n$.

Use Observed Data: Decisioning while Learning



Resources allocated at each update point is proportional to the number of customers already arrived.

Theorem: Let the orders come randomly and let

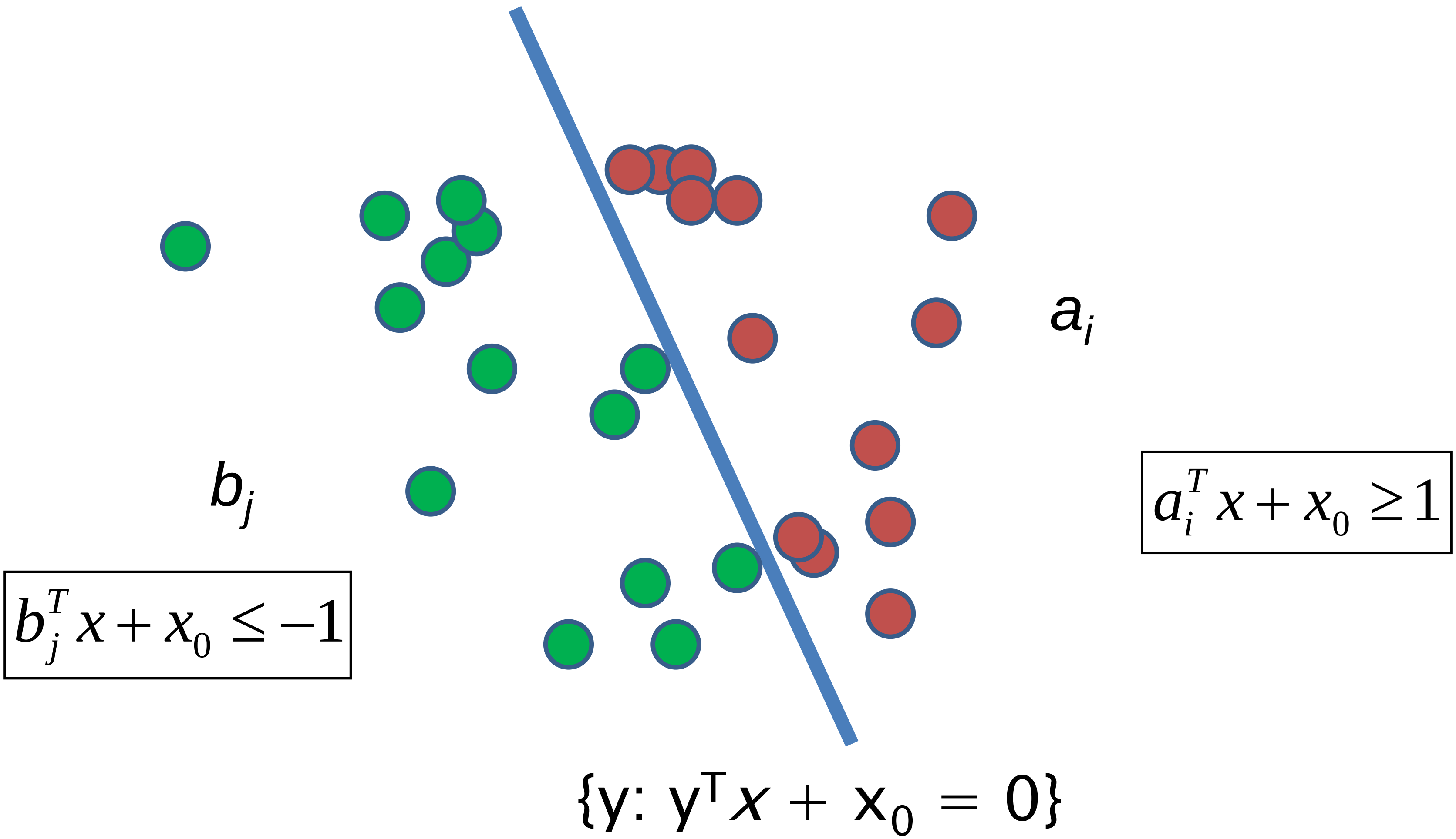
$$\min_i \{ b_i \} \geq m\log(n)/\varepsilon^2.$$

Then
the expected online revenue ≥ (1 – ε) the offline revenue.

Theorem: On the other hand, if $\min_i \{ b_i \} < \log(m)/\varepsilon^2$.
then no mechanism/algorithm can achieve the (1 – ε) guarantee.

LP Project: Second-Order Support Vector Machine (SVM)

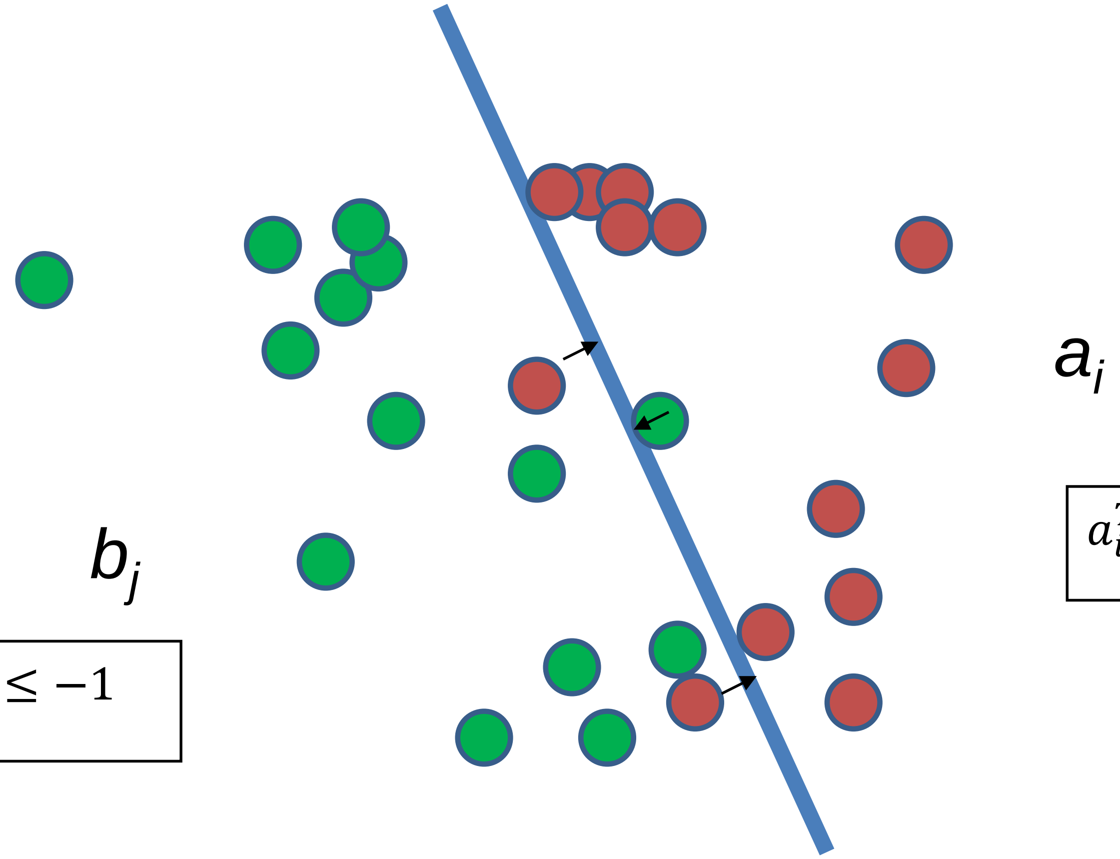
Given two sets of points (rad and blue), find a line/plane to separate them,



x is the normal direction or slope vector and x_0 is the intersect

Find a line to **strictly** separate greens and reds

Minimize Error if Strict Separation is Impossible I

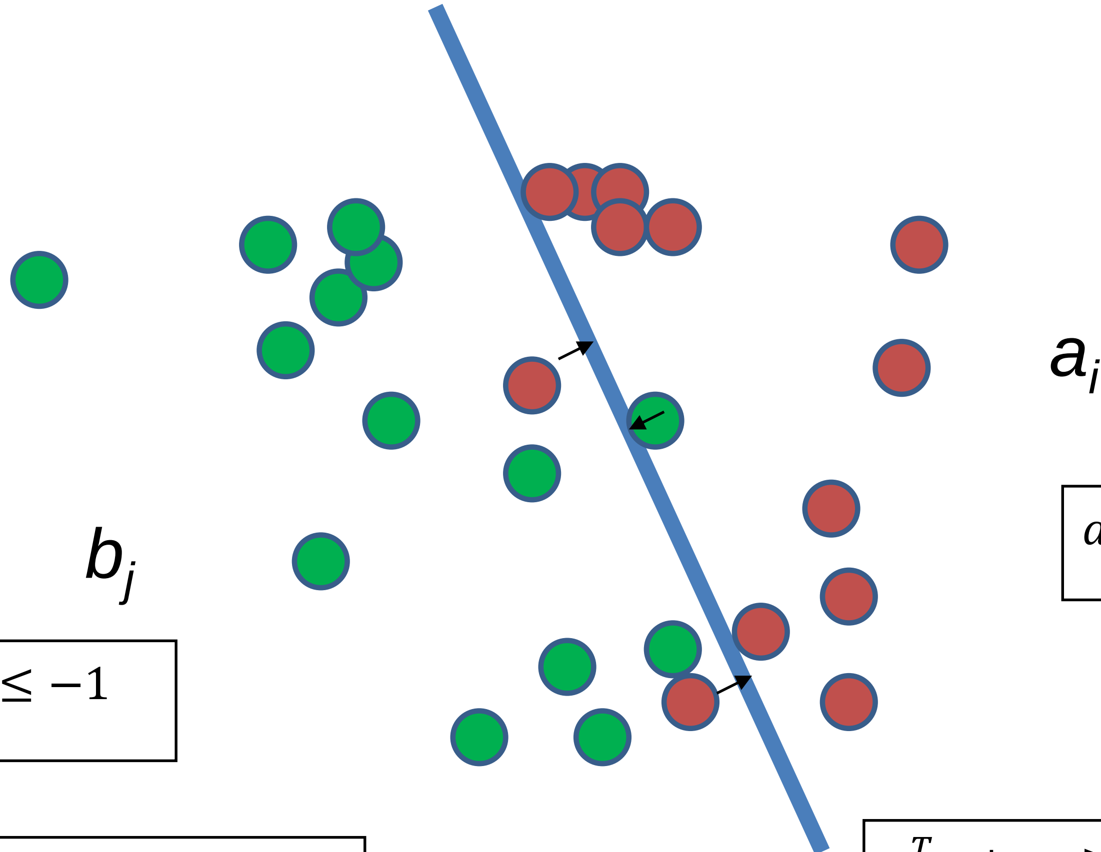


$a_i^T x + x_0 \geq 1$

$b_j^T x + x_0 \leq -1$

$$\text{minimize} \left\{ \textstyle\sum_i \max\left(1 - a_i^T x - x_0 , 0\right) + \sum_j \max(b_j^T x + x_0 + 1, 0) \right\}$$

Minimize Error if Strict Separation is Impossible II



$$a_i^T x + x_0 \geq 1$$

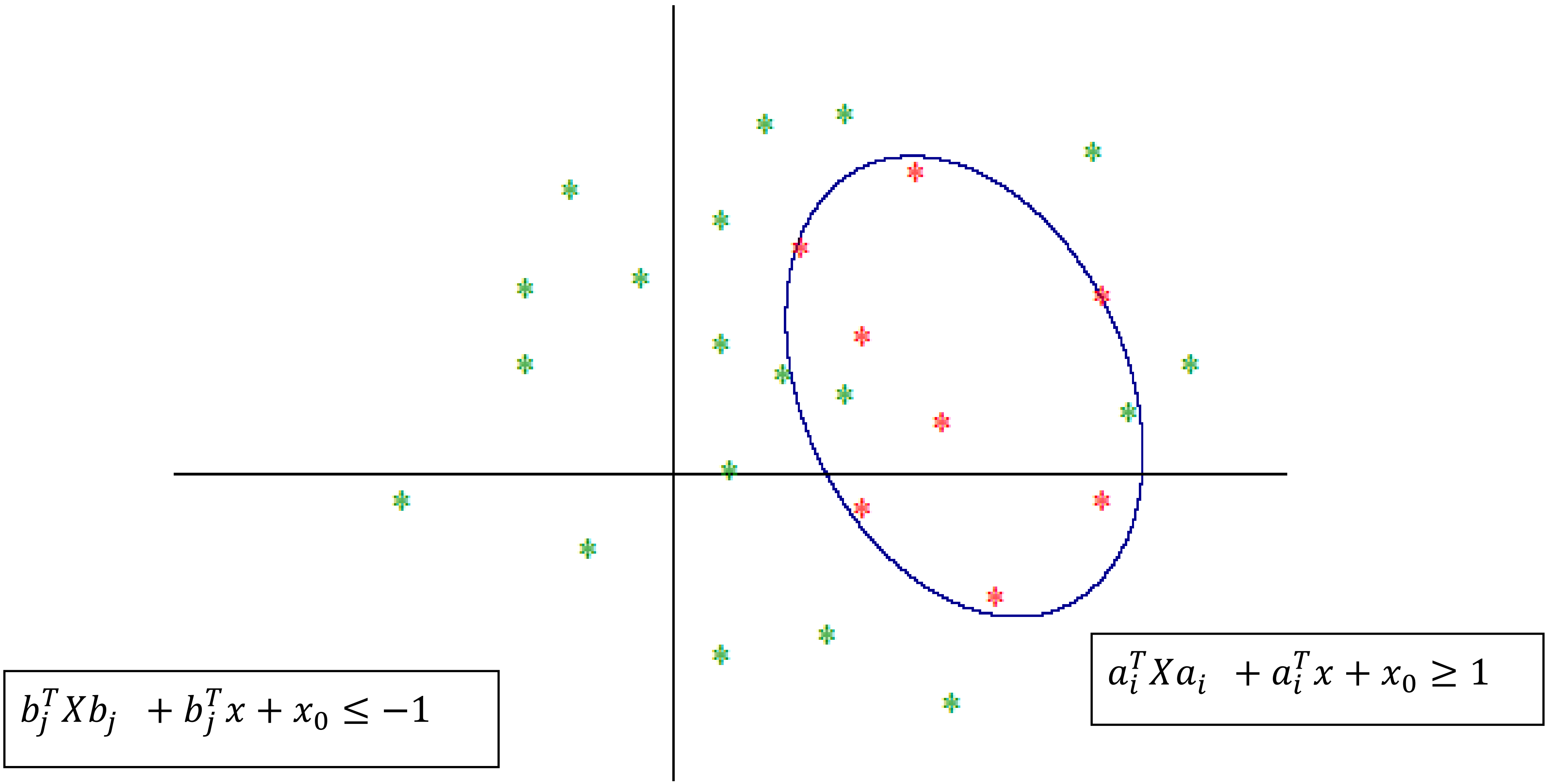
$$a_i^T x + x_0 \geq 1 - \delta'_i, \delta'_i \geq 0$$

$$b_j^T x + x_0 \leq -1$$

$$b_j^T x + x_0 \leq -1 + \delta"_j, \delta"_j \geq 0$$

$$\text{minimize} \quad \sum_i \delta'_i + \sum_j \delta"_j$$

SVM with Correlation Information ?



X is the normal direction or slope vector of Second Moment of Data-Points
This would produce a Quadratic Curve
If *X* is required to be Positive Definite, the Curve becomes an Ellipsoid.