

More Duality Concepts/Applications and Course Summaries

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<https://web.stanford.edu/class/msande211x/handout.shtml>

Chapter 3.1-3.5

Two-Person Zero-Sum Matrix Game

$$\begin{pmatrix} 3 & -1 & -3 \\ -3 & 1 & 4 \end{pmatrix} = P$$

P is the payoff matrix of a two-person, "Column" and "Row", zero-sum game.
Player Column chooses column(s) to maximize the payoff to Column
Player Row chooses row(s) to minimize the payoff to Column

Pure Strategy: Each player chooses a single column (row).

Mixed or Randomized Strategy: Each player randomly chooses columns (rows) strategies with a fixed probability distribution.

Nash Equilibrium: No player can alter its probability distribution to achieve better expected payoff.

Two-Person Zero-Sum Matrix Game II

$$\begin{pmatrix} 3 & -1 & -3 \\ -3 & 1 & 4 \end{pmatrix}$$

Player Column Player: probabilities x_1 to choose column 1, x_2 to choose column 2, and x_3 to choose column 3. Then the expected payoff is

$$\begin{array}{ll} 3x_1 - x_2 - 3x_3 & \text{if Player Row chooses row 1} \\ -3x_1 + x_2 + 4x_3 & \text{if Player Row chooses row 2} \end{array}$$

Thus, Player Column would

$$\begin{array}{ll} \text{maximize}_{(x_1, x_2, x_3)} & \min\{3x_1 - x_2 - 3x_3, -3x_1 + x_2 + 4x_3\} \\ \text{s.t.} & x_1 + x_2 + x_3 = 1, (x_1, x_2, x_3) \geq 0 \end{array}$$

which can be cast as a linear program

$$\begin{array}{llll} \text{maximize}_{(x_1, x_2, x_3, v)} & & v & \\ \text{s.t.} & -3x_1 + x_2 + 3x_3 + v \leq 0 & & y_1 \\ & 3x_1 - x_2 - 4x_3 + v \leq 0 & & y_2 \\ & x_1 + x_2 + x_3 = 1, & & u \\ & (x_1, x_2, x_3) \geq 0 & & \end{array}$$

Two-Person Zero-Sum Matrix Game III

$$\begin{pmatrix} 3 & -1 & -3 \\ -3 & 1 & 4 \end{pmatrix}$$

Then, the dual of the linear program

$$\begin{aligned} & \text{minimize}_{(y_1, y_2, u)} && u \\ & \text{s.t.} && u - (3y_1 - 3y_2) \geq 0 \\ & && u - (-y_1 + y_2) \geq 0 \\ & && u - (-3y_1 + 4y_2) \geq 0 \\ & && y_1 + y_2 = 1, (y_1, y_2) \geq 0 \end{aligned}$$

Interpretations:

Player Row: probabilities y_1 to choose row 1, y_2 to choose row 2. Then the expected payoff to Player Column is

$$\begin{aligned} & 3y_1 - 3y_2 && \text{if Player Column chooses column 1} \\ & -y_1 + y_2 && \text{if Player Column chooses column 2} \\ & -3y_1 + 4y_2 && \text{if Player Column chooses column 3;} \end{aligned}$$

and Player Row does

$$\begin{aligned} & \text{minimize}_{(y_1, y_2)} && \max\{3y_1 - 3y_2, -y_1 + y_2, -3y_1 + 4y_2\} \\ & \text{s.t.} && y_1 + y_2 = 1, (y_1, y_2) \geq 0 \end{aligned}$$

Consider a Simplified MDP-RL Problem (Maze-Run)

$$\max y_0 + y_1 + y_2 + y_3 + y_4 + y_5$$

$$\text{s.t. } y_5 \leq 0 + \gamma y_5$$

$$y_4 \leq 1 + \gamma y_5$$

$$y_3 \leq 0 + \gamma y_4$$

$$y_3 \leq 0 + \gamma y_5$$

$$y_2 \leq 0 + \gamma y_3$$

$$y_2 \leq 0 + \gamma(0.5y_4 + 0.5y_5)$$

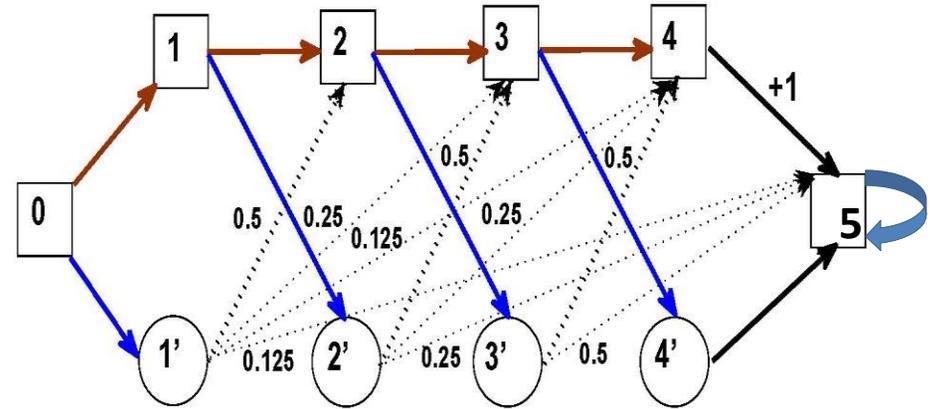
$$y_1 \leq 0 + \gamma y_2$$

$$y_1 \leq 0 + \gamma(0.5y_3 + 0.25y_4 + 0.25y_5) \quad \mathbf{y^*_0 = y^*_1 = y^*_2 = y^*_3 = y^*_5 = 0}$$

$$y_0 \leq 0 + \gamma y_1$$

$$\mathbf{y^*_4 = 1}$$

$$y_0 \leq 0 + \gamma(0.5y_2 + 0.25y_3 + 0.125y_4 + 0.125y_5)$$



- y_i : expected overall cost if starting from State i .
- State 4 is a trap
- State 5 is the destination
- Each other state has two options: Go directly to the next state OR a short-cut go to other states with uncertainties

Physical Interpretation of the Maze-Run Dual

$$\max y_0 + y_1 + y_2 + y_3 + y_4 + y_5$$

$$\text{s.t. } y_5 \leq 0 + \gamma y_5 \quad (x_5)$$

$$y_4 \leq 1 + \gamma y_5 \quad (x_4)$$

$$y_3 \leq 0 + \gamma y_4 \quad (x_{3r})$$

$$y_3 \leq 0 + \gamma y_5 \quad (x_{3b})$$

$$y_2 \leq 0 + \gamma y_3 \quad (x_{2r})$$

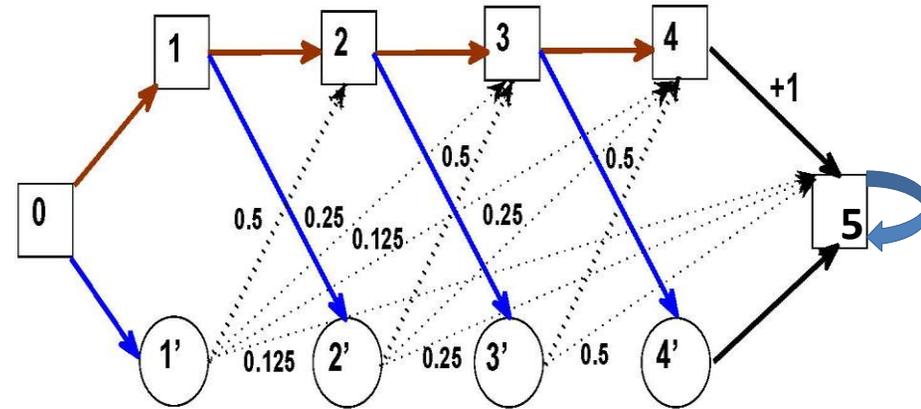
$$y_2 \leq 0 + \gamma(0.5y_4 + 0.5y_5) \quad (x_{2b})$$

$$y_1 \leq 0 + \gamma y_2 \quad (x_{1r})$$

$$y_1 \leq 0 + \gamma(0.5y_3 + 0.25y_4 + 0.25y_5) \quad (x_{1b})$$

$$y_0 \leq 0 + \gamma y_1 \quad (x_{0r})$$

$$y_0 \leq 0 + \gamma(0.5y_2 + 0.25y_3 + 0.125y_4 + 0.125y_5) \quad (x_{0b})$$



x_j represents
(discounted) how many
expected times
(frequency) actions j
being taken in a policy.

The Dual of the Maze Example

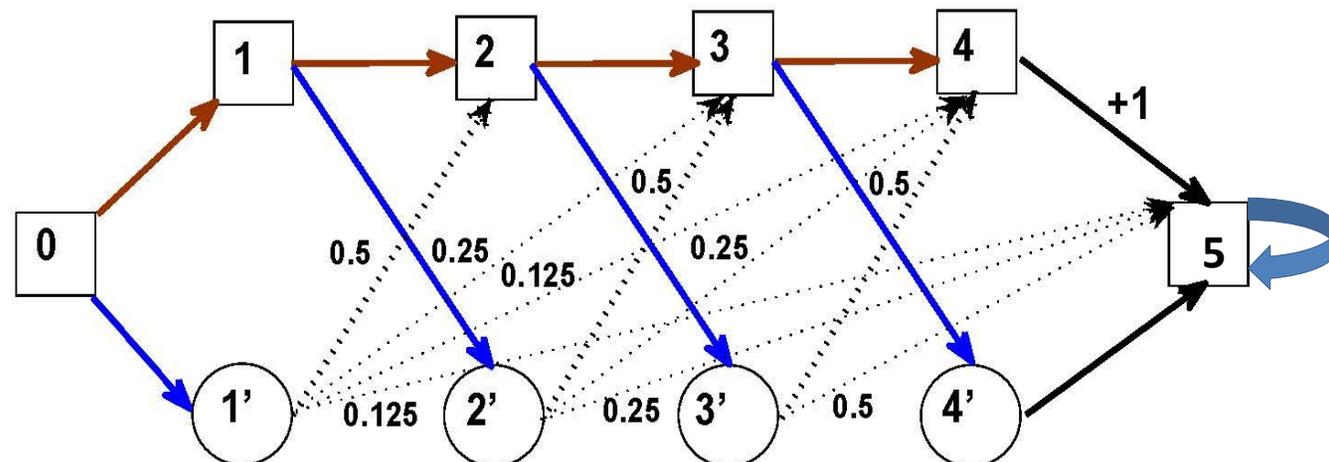
x:	(0r)	(0b)	(1r)	(1b)	(2r)	(2b)	(3r)	(3b)	(4)	(5)	b
c:	0	0	0	0	0	0	0	0	1	0	
(0)	1	1	0	0	0	0	0	0	0	0	1
(1)	$-\gamma$	0	1	1	0	0	0	0	0	0	1
(2)	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	0	0	1
(3)	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	1
(4)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	0	1
(5)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	0	$-\gamma$	$-\gamma$	$1-\gamma$	1

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = e, (y) \\ & x \geq 0. \end{aligned}$$

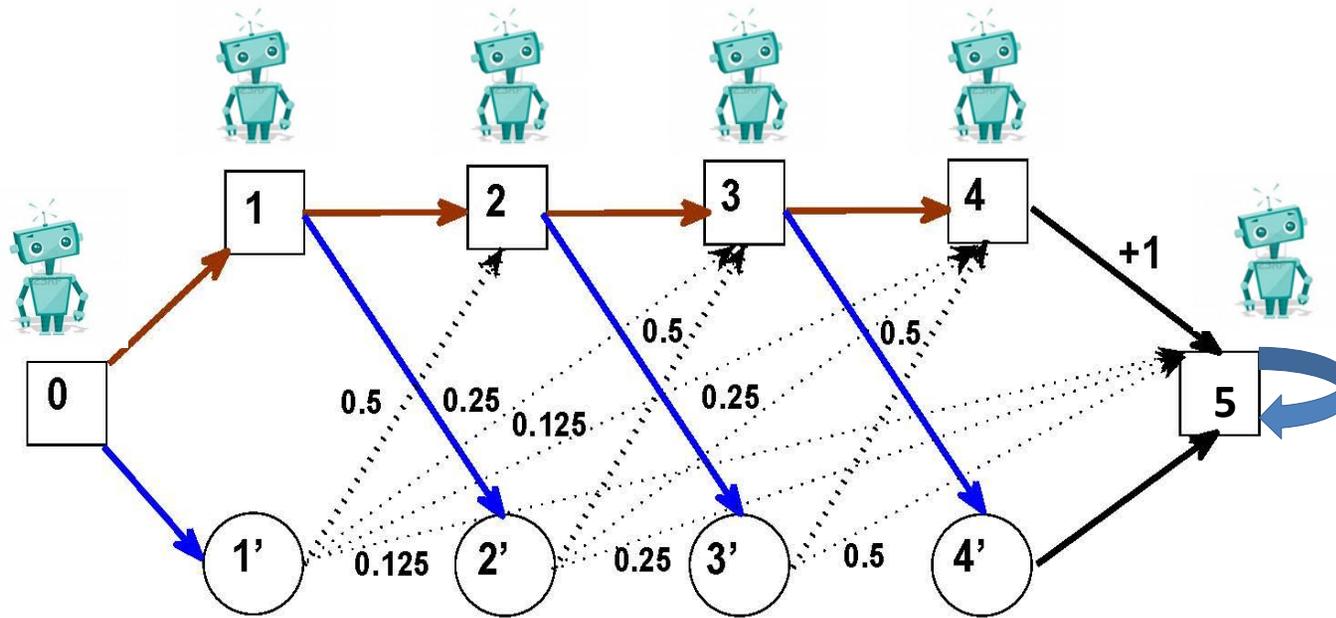
The optimal dual solution is

$$x_{0r}^* = 1, x_{1r}^* = 1 + \gamma, x_{2r}^* = 1 + \gamma + \gamma^2, x_{3b}^* = 1 + \gamma + \gamma^2 + \gamma^3, x_4^* = 1,$$

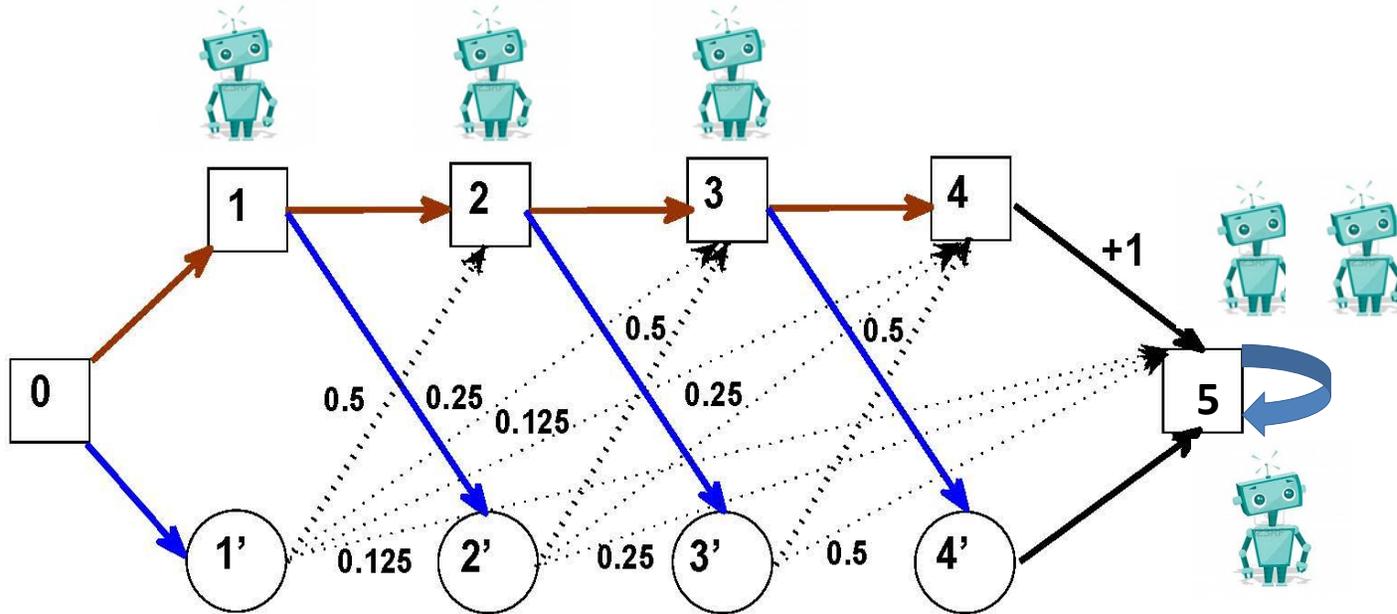
$$x_5^* = \frac{1 + 2\gamma + \gamma^2 + \gamma^3 + \gamma^4}{1 - \gamma}.$$



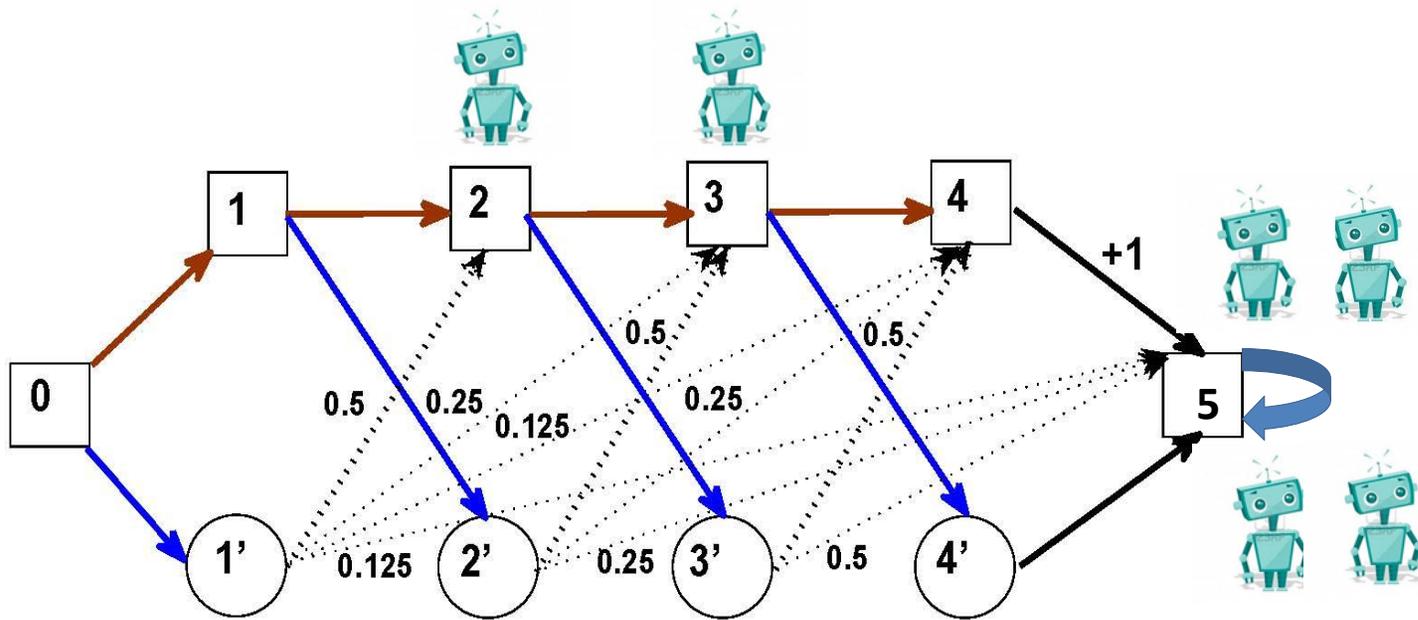
Time 1



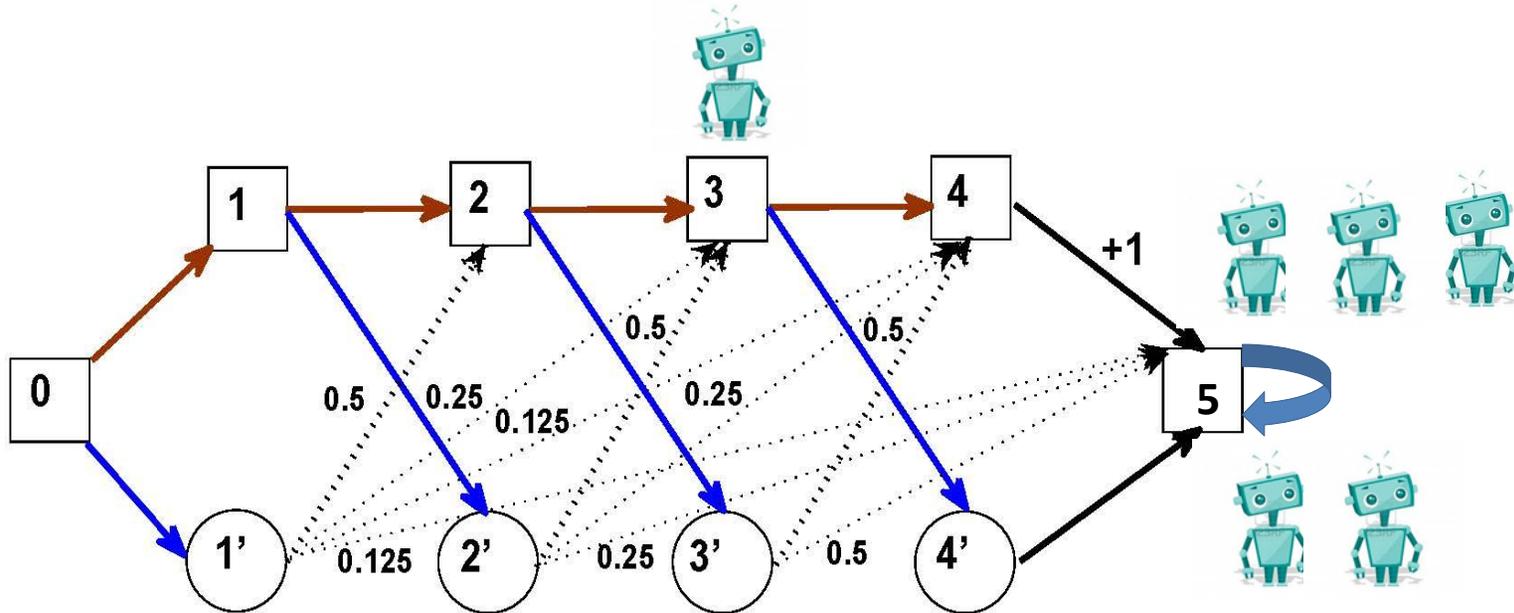
Time 2



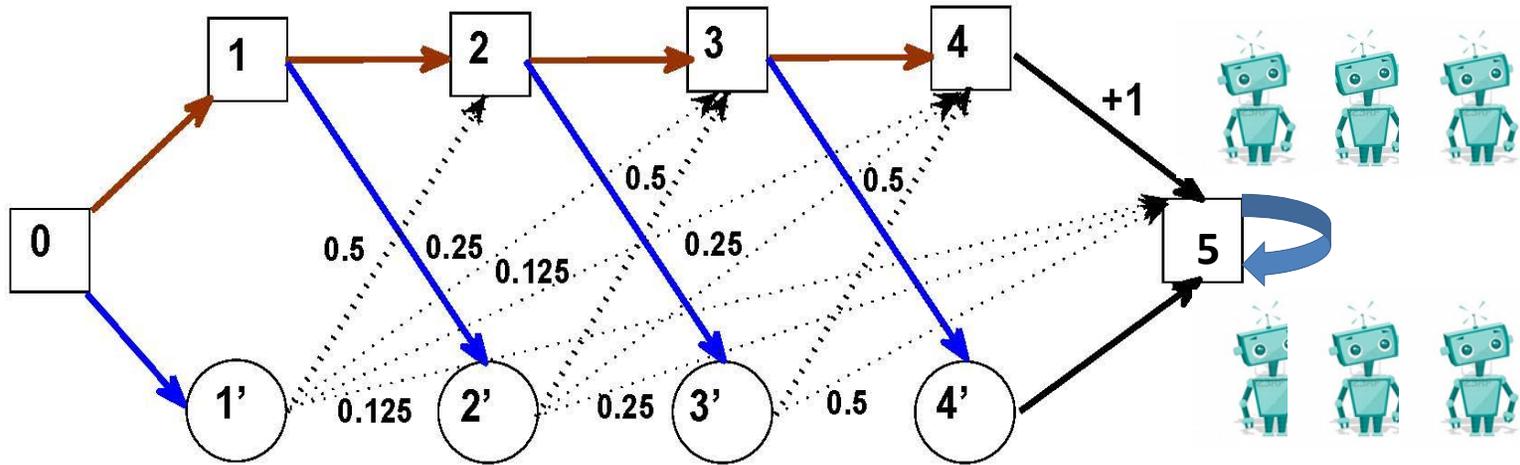
Time 3



Time 4



Time 5



Recall the optimal dual solution values are:

$$x_{0r}^* = 1, x_{1r}^* = 1 + \gamma, x_{2r}^* = 1 + \gamma + \gamma^2, x_{3b}^* = 1 + \gamma + \gamma^2 + \gamma^3, x_4^* = 1,$$

$$x_5^* = \frac{1 + 2\gamma + \gamma^2 + \gamma^3 + \gamma^4}{1 - \gamma}.$$

LP Duality Theorem

Theorem 1 (*Weak duality theorem*) Let both primal feasible region F_p and dual feasible region F_d be non-empty. Then,

$$\mathbf{c}^T \mathbf{x} \geq \mathbf{b}^T \mathbf{y} \quad \text{for all} \quad \mathbf{x} \in F_p, \mathbf{y} \in F_d.$$

Proof: $\mathbf{c}^T \mathbf{x} - \mathbf{b}^T \mathbf{y} = \mathbf{c}^T \mathbf{x} - (\mathbf{A}\mathbf{x})^T \mathbf{y} = \mathbf{x}^T (\mathbf{c} - \mathbf{A}^T \mathbf{y}) = \mathbf{x}^T \mathbf{r} \geq 0.$

This theorem shows that a feasible solution to either problem yields a **bound** on the value of the other problem. We call $\mathbf{c}^T \mathbf{x} - \mathbf{b}^T \mathbf{y}$ the **duality gap**.

If the duality gap is zero, then \mathbf{x} and \mathbf{y} are **optimal** for the primal and dual, respectively! Is the reverse true?

LP Duality Theorem continued

Proved by the Simplex Method

Theorem 2 (Strong duality theorem) Let both primal feasible region F_p and dual feasible region F_d be non-empty. Then, $\mathbf{x}^* \in F_p$ is optimal for (LP) and $\mathbf{y}^* \in F_d$ is optimal for (LD) if and only if the duality gap $\mathbf{c}^T \mathbf{x}^* - \mathbf{b}^T \mathbf{y}^* = 0$ (no need for technical assumptions).

Corollary If (LP) and (LD) both have feasible solutions then both problems have optimal solutions and the optimal objective values of the objective functions are equal.

If one of (LP) or (LD) has no feasible solution, then the other is either *unbounded* or has no feasible solution. If one of (LP) or (LD) is unbounded then the other has no feasible solution.

Possible Combination of Primal and Dual

Dual \ Primal	F-B	F-UB	IF
F-B			
F-UB			
IF			

$$\begin{array}{ll}
 \min & -x_1 - x_2 \\
 \text{s.t.} & x_1 - x_2 = 1 \\
 & -x_1 + x_2 = 1 \\
 & x_1, x_2 \geq 0
 \end{array}$$

$$\begin{array}{ll}
 \max & y_1 + y_2 \\
 \text{s.t.} & y_1 - y_2 \leq -1 \\
 & -y_1 + y_2 \leq -1
 \end{array}$$

Application: World Cup Trading Markets

- Create a market for World Cup security exchange
 - Five securities in a market for open trading at fixed prices and pay-offs, and **short is allowed**
 - We'd like to decide how many shares to purchase or sell to maximize the worst case pay-off when the game is realized.

Security	Price π	Share Limit q	Argentina	Brazil	Italy	Germany	France
1	\$0.75	∞	\$1	\$1	\$1		
2	\$0.35	∞				\$1	
3	\$0.40	∞	\$1		\$1		\$1
4	\$0.95	∞	\$1	\$1	\$1	\$1	
5	\$0.75	∞		\$1		\$1	

Portfolio Optimization Model

No share limit, and short is allowed:

$$\max \quad s - .75x_1 - .35x_2 - .4x_3 - .95x_4 - .75x_5$$

$$\text{s.t.} \quad s - x_1 - x_3 - x_4 \leq 0$$

$$s - x_1 - x_4 - x_5 \leq 0$$

$$s - x_1 - x_3 - x_4 \leq 0$$

$$s - x_2 - x_4 - x_5 \leq 0$$

$$s - x_3 \leq 0$$

all variables free

Is this (primal) problem feasible?

Is this problem unbounded? – Check if the dual is feasible or not.

Dual of the Portfolio Optimization Model

$$\min \mathbf{0}^T \bullet \mathbf{p}$$

$$\text{s.t.} \quad p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

$$p_1 + p_2 + p_3 = .75$$

$$p_4 = .35$$

$$p_1 + p_3 + p_5 = .4$$

$$p_1 + p_2 + p_3 + p_4 = .95$$

$$p_2 + p_4 = .75$$

$$p_1, p_2, p_3, p_4, p_5 \geq 0$$

P : the state prices

Is the dual feasible? Arbitrage if it is not!

Show infeasibility by inspection

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

$$p_1 + p_2 + p_3 = .75$$

$$p_4 = .35$$

$$p_1 + p_3 + p_5 = .4$$

$$p_1 + p_2 + p_3 + p_4 = .95$$

$$p_2 + p_4 = .75$$

$$p_1, p_2, p_3, p_4, p_5 \geq 0$$

Manipulate the constraints to find a contradiction!

On-Line Retailer Linear Programming

- Off-line Problem is an (0,1) linear program that can be relaxed as LP
- But now trader/Bidders come one by one **sequentially**,
- The retailer has to make the decision **as soon as an order arrives** with the arrived combinatorial order/bid (\mathbf{a}_k, π_k)
- The retailer faces a dilemma:
 - **To sell or not to sell – this is the decision**
- Optimal Policy or Online Algorithm?

$$\begin{aligned} \max \quad & \sum_{j=1}^n \pi_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \forall i = 1, \dots, m \\ & x_j = \{0 \text{ or } 1\} \quad \forall j = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{j=1}^n \pi_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \forall i = 1, \dots, m \\ & 0 \leq x_j \leq 1 \quad \forall j = 1, \dots, n \end{aligned}$$

Off-Line LP Relaxation

CSC of Off-Line Retailer Linear Programming

- Let the optimal solution be \mathbf{x}^* and the optimal shadow piece be \mathbf{y}^*

- Then from the CSC conditions:

$$x_j^* = 1 \text{ if } \pi_j > \mathbf{a}_j^T \mathbf{y}^*$$

$$x_j^* = 0 \text{ if } \pi_j < \mathbf{a}_j^T \mathbf{y}^*$$

$$x_j^* = \text{fraction} \text{ if } \pi_j = \mathbf{a}_j^T \mathbf{y}^*$$

- If we know \mathbf{y}^* , the online decision would be easy!

$$\begin{array}{ll} \max & \sum_{j=1}^n \pi_j x_j \\ \text{s.t.} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \forall i = 1, \dots, m \\ & 0 \leq x_j \leq 1 \quad \forall j = 1, \dots, n \end{array}$$

Off-Line LP Relaxation

Online Algorithm and Price-Mechanism

- Learn “ideal” itemized optimal prices
- Use the prices to price each bid
- Accept if it is a over bid, and reject otherwise

Bid #	\$100	\$30	Inventory	Price?
Decision	x1	x2					
Pants	1	0	100	45
Shoes	1	0				50	45
T-Shirts	0	1				500	10
Jackets	0	0				200	55
Hats	1	1	1000	15

Such ideal prices exist, and they are shadow/dual prices of the offline LP

How to Learn the Shadow Prices Sequentially?

- **Sequential Linear Programming Mechanism (SLPM)**
 - Solving the LP based on immediately past several periods' data and use the resulted optimal shadow prices to make decision for the next period orders; and repeat when the current period is over.
- The **shadow prices** are **updated periodically** and being used to make online decisions for the next period.

Wait for Data from 1 to εn

- Set $x_j=0$ for $j=1,\dots,\varepsilon n$.
- Solve LP:
- Let p^1 be the **optimal shadow price vector** and use it to make online decision for orders from $\varepsilon n+1$ to $2\varepsilon n$.

$$\begin{array}{ll} \max & \sum_{j=1}^{\varepsilon n} \pi_j x_j \\ \text{s.t.} & \sum_{j=1}^{\varepsilon n} a_{ij} x_j \leq \varepsilon b_i \quad \forall i \\ & 0 \leq x_j \leq 1 \quad \forall j \end{array}$$

Now Use All Data from 1 to $2\epsilon n$

- Now solve LP:

$$\begin{array}{ll} \max & \sum_{j=1}^{2\epsilon n} \pi_j x_j \\ \text{s.t.} & \sum_{j=1}^{2\epsilon n} a_{ij} x_j \leq 2\epsilon b_i \quad \forall i \\ & 0 \leq x_j \leq 1 \quad \forall j \end{array}$$

- Let \mathbf{p}^2 be the **optimal shadow price vector** and use it to make online decision for orders from $2\epsilon n + 1$ to $4\epsilon n$.

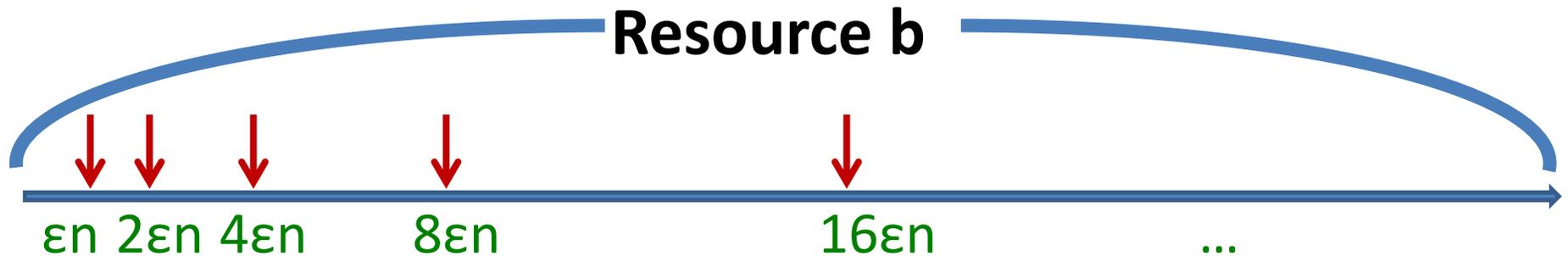
Now Use All Data from 1 to $4\epsilon n$

- Now solve LP:

$$\begin{array}{ll} \max & \sum_{j=1}^{4\epsilon n} \pi_j x_j \\ \text{s.t.} & \sum_{j=1}^{4\epsilon n} a_{ij} x_j \leq 4\epsilon b_i \quad \forall i \\ & 0 \leq x_j \leq 1 \quad \forall j \end{array}$$

- Let \mathbf{p}^3 be the **optimal shadow price vector** and use it to make online decision for orders from $4\epsilon n + 1$ to $8\epsilon n$.

Use Observed Data: Decisioning while Learning



Resources allocated at each update point is proportional to the number of customers already arrived.

Theorem: Let the orders come randomly and let

$$\min_i \{ b_i \} \geq m \log(n) / \epsilon^2.$$

Then

the expected online revenue $\geq (1 - \epsilon)$ the offline revenue.

On the other hand, if $\min_i \{ b_i \} < \log(m) / \epsilon^2$.

then no mechanism/algorithm can achieve the $(1 - \epsilon)$ guarantee.

Summary: The Geometry of LPs

- Understand the geometrical interpretation of LPs and associated intuition
 - Plot a feasible region in 2D
 - Plot the optimal contour lines or curves
 - Identify the corner optimal point corner point or BFS.
 - Understand the geographical interpretation of the various LP terms – (e.g. active constraints, basic variables, multiple optimizers, infeasibility, unboundedness, etc.)
 - The corner (feasible) solutions of the feasible region
- The Simplex method: move from corner to corner!

Summary: Theory of Linear Programming

An LP (or any optimization) problem falls in one of three cases:

- Problem is *infeasible*: Feasible region is empty.
- Problem is *unbounded*: Feasible region is unbounded towards the optimizing direction -- no minimizer
- Problem is *feasible* and *bounded*: then there exists an optimal point; an optimal point is on the boundary of the feasible region; and there is always at least one optimal corner point (if the feasible region has a corner point).

When the problem is feasible and bounded,

- There may be a unique optimizer or infinitely many multiple optimizers (alternative optimizers, adjacent or not).
- If a corner point or BFS is not “worse” than all its neighbor corners, then it is optimal.

Shadow Price (Lagrange Multiplier, Dual Variable)

- Each constraint is associated with a **shadow price, also called Lagrange multiplier or dual variable**
- They are used to **certify** whether or not a feasible solution is optimal.
- At an optimal solution, all **inactive** constraints have zero-valued Lagrange multiplier (called complementarity)
- At optimality, the Lagrange multiplier on a given active constraint is the **rate of change** in the **optimal value (OV)** as the RHS of the constraint increases with all other data held fixed.
- The **reduced cost** can be viewed as the Lagrange multiplier of the nonnegative constraint; a BFS is minimal if all reduced costs become nonnegative.

The Primal and Dual Pair

- Every optimization problem is associated with another optimization problem called **dual** (the original problem is called **primal**).
- Every **variable** of the dual is the Lagrange multiplier associated with a **constraint** in the primal.
- The dual is **max** (**min**) if the primal is **min** (**max**)
- The dual is a **convex** optimization problem. Moreover, the two optimal objective values are equal if the primal is also a **convex** optimization problem (under mild technical assumptions).
- The **optimal** solution of the dual is the optimal/KKT Lagrange multiplier or **shadow price** vector of the RHS **b** of the primal, or $\nabla f^*(b) = y^*$
- The optimal primal and dual solution meet the FONC or **KKT conditions**.
- Many applications using the duality: **zero-sum and production games**, etc.

The LP Optimality Condition

Check if a pair of primal \mathbf{x} and dual \mathbf{y} , with slack \mathbf{r} , is optimal:

$$\left\{ \begin{array}{l} (\mathbf{x}, \mathbf{y}, \mathbf{r}) \in (R_+^n, R^m, R_+^n) : \\ \begin{array}{|l} \mathbf{c}^T \mathbf{x} - \mathbf{b}^T \mathbf{y} = \mathbf{0} \\ \mathbf{A} \mathbf{x} = \mathbf{b} \\ \mathbf{A}^T \mathbf{y} + \mathbf{r} = \mathbf{c} \end{array} \end{array} \right\},$$

which is a system of linear inequalities and equations. Thus it is easy to verify whether or not a pair $(\mathbf{x}, \mathbf{y}, \mathbf{r})$ is optimal by a computer.

These conditions can be classified as

- Primal Feasibility,
- Dual Feasibility, and
- Zero Duality Gap.