The Dual Problem Interpretations and Applications

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Chapter 3.1-3.5

The Economic Interpretation of the Production Dual

Primal

$$\begin{array}{lll} \max & x_1 + 2x_2 & \\ \text{s.t.} & x_1 & \leq 1 \\ & \quad\quad x_2 & \leq 1 \\ & x_1 + x_2 & \leq 1.5 \\ & x_1, \quad x_2 & \geq 0 \end{array}$$

Dual

$$\begin{array}{lll} \min & y_1 + y_2 + 1.5y_3 & \\ \text{s.t.} & y_1 \quad\quad + y_3 & \geq 1 \\ & \quad\quad y_2 + y_3 & \geq 2 \\ & y_1, \quad y_2, \quad y_3 & \geq 0 \end{array}$$

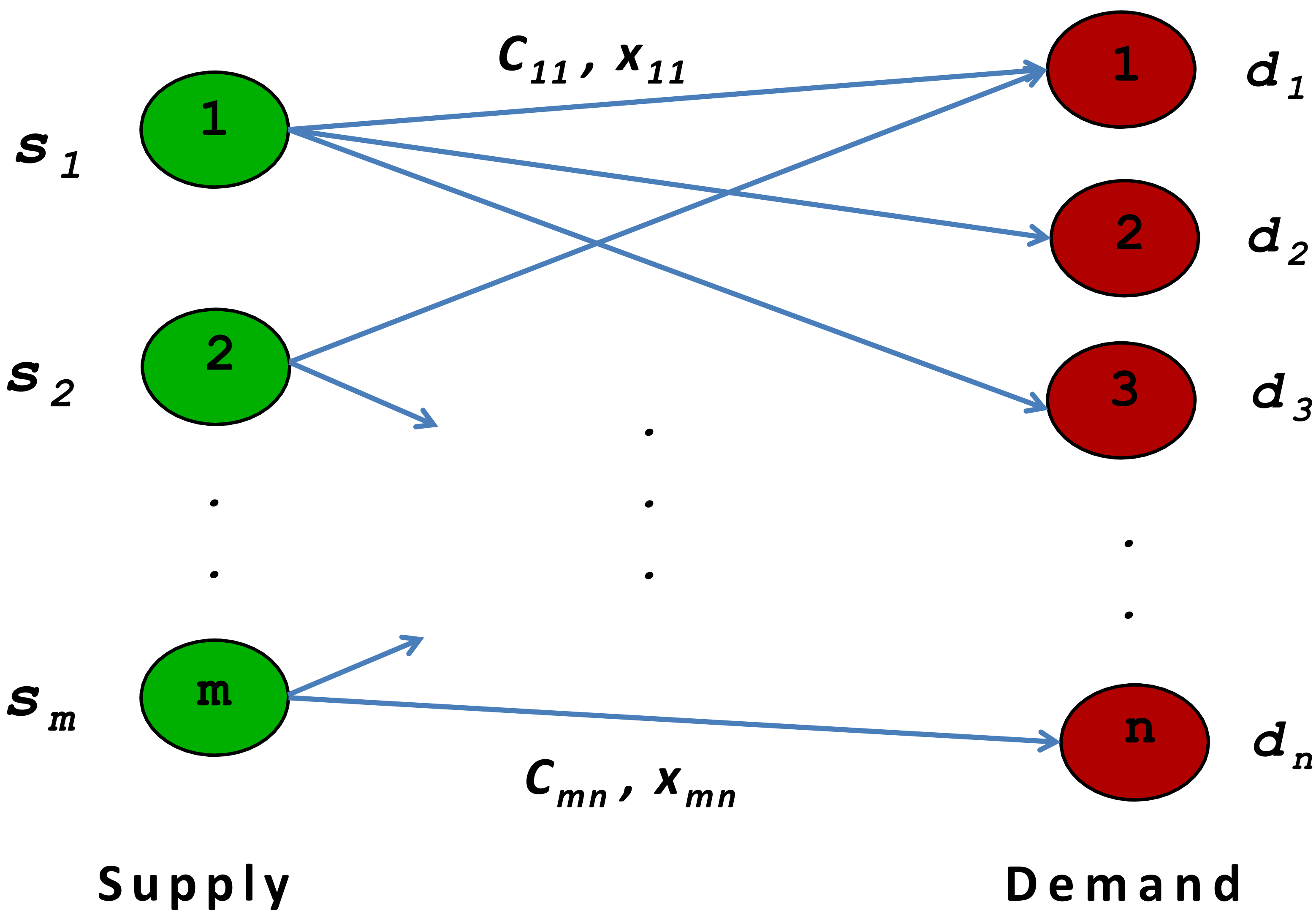
$\max\ \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$

$\min\ \mathbf{b}^T\mathbf{y}$ s.t. $A^T\mathbf{y} \geq \mathbf{c},\ \mathbf{y} \geq \mathbf{0}$

Acquisition Pricing:

- $\boldsymbol{y}$: prices of the resources
- $A^T\boldsymbol{y} \geq \boldsymbol{c}$: the prices are competitive for each product
- $\min\ \boldsymbol{b}^T\boldsymbol{y}$: minimize the total liquidation cost

The Transportation Dual



The Transportation Example

	1	2	3	4	Supply
1	12	13	4	6	500 u_1
2	6	4	10	11	700 u_2
3	10	9	12	4	800 u_3
Demand	400 v_1	900 v_2	200 v_3	500 v_4	20000

The Dual Interpretation: New Pricing Mechanism

Primal

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} = s_i, & \forall\, i = 1,\ldots,m \\
 & \sum_{i=1}^{m} x_{ij} = d_j, & \forall\, j = 1,\ldots,n \\
 & x_{ij} \ge 0, & \forall\, i, j
\end{array}
$$

Dual

$$
\begin{array}{lll}
\max & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j & \\
\text{s.t.} & u_i + v_j \le c_{ij}, & \forall\, i, j
\end{array}
$$

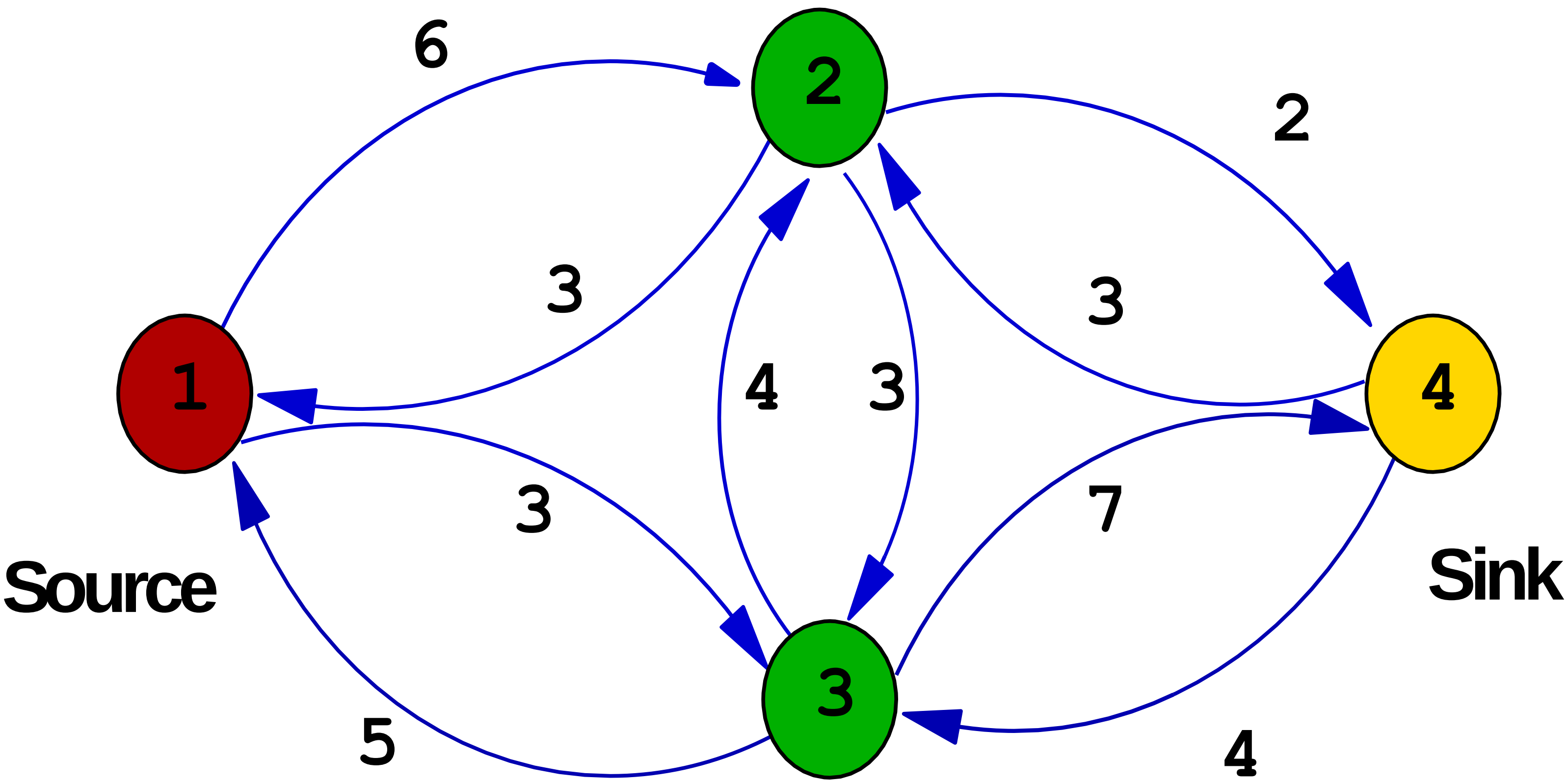
Shipping Company's new charge scheme:

u_i: supply site unit charge

v_j: demand site unit charge

$u_i + v_j \le c_{ij}$: competitiveness

Look at a Max-Flow Problem



The Primal Formulation

Let x_{ij} be the flow rate from node i to node j. Then the problem can be formulated as

$$
\begin{array}{lllc}
\max & x_{41} & & \\
\text{s.t.} & x_{21}+x_{31}+x_{41}-x_{12}-x_{13} & =0, & y_1 \\
 & x_{12}+x_{32}+x_{42}-x_{21}-x_{23}-x_{24} & =0, & y_2 \\
 & x_{13}+x_{23}+x_{43}-x_{31}-x_{32}-x_{34} & =0, & y_3 \\
 & x_{24}+x_{34}-x_{41}-x_{42}-x_{43} & =0, & y_4 \\
 & x_{ij} & \le k_{ij},\ \forall (i,j)\in A, & z_{ij} \\
 & x_{ij} & \ge 0,\ \forall (i,j)\in A. &
\end{array}
$$

Corresponding
Dual variables

The Dual of Max-Flow: the Min-Cut Problem

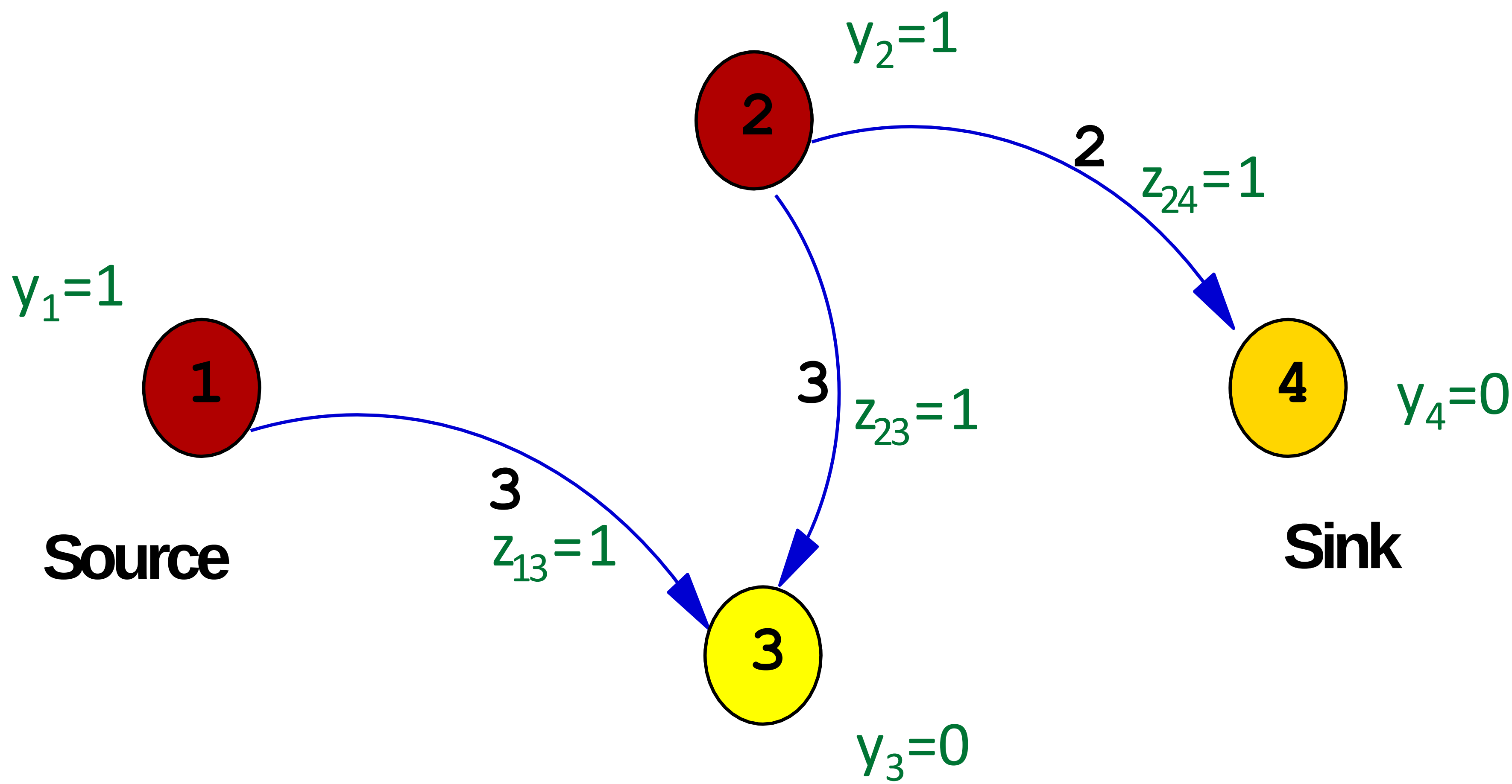
$$
\begin{array}{lll}
\min & \sum_{(i,j)\in A} k_{ij} z_{ij} & \\
\text{s.t.} & y_1 - y_4 = 1, & x_{41} \\
& -y_1 + y_2 + z_{12} \geq 0, & x_{12} \\
& -y_1 + y_3 + z_{13} \geq 0, & x_{13} \\
& \ldots & \ldots \\
& -y_2 + y_4 + z_{24} \geq 0, & x_{24} \\
& -y_3 + y_4 + z_{34} \geq 0, & x_{34} \\
& z_{ij} \geq 0, \ \forall (i,j) \in A. &
\end{array}
$$

Corresponding Primal variables: x_{41}, x_{12}, x_{13}, …, x_{24}, x_{34}

y_i: node potential value; wlog set $y_4 = 0$ so that $y_1 = 1$ and at optimality for all other y_i:

$$
y_i = \begin{cases} 1 & \text{if } i \text{ is on the source side} \\ 0 & \text{if } i \text{ is not in the source side} \end{cases}
\quad \text{and} \quad
z_{ij} = \begin{cases} 1, & \text{if } y_i = 1 \text{ and } y_j = 0 \\ 0 & \text{otherwise} \end{cases}
$$

The Min-Cut Solution: Min-Cut Value=8

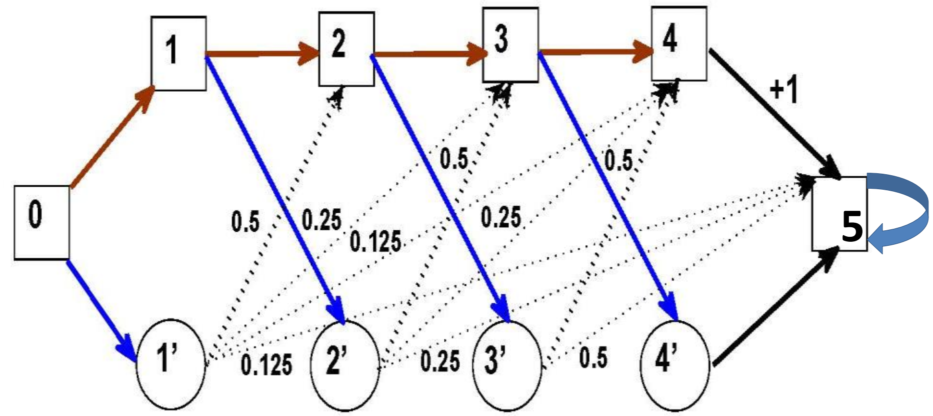


All other $z_{ij}=0$

Consider a Simplified MDP-RL Problem (Maze-Run)

$$\max\ y_0 + y_1 + y_2 + y_3 + y_4 + y_5$$

$$\begin{aligned}
\text{s.t.}\quad & y_5 \le 0 + \gamma y_5 \\
& y_4 \le 1 + \gamma y_5 \\
& y_3 \le 0 + \gamma y_4 \\
& y_3 \le 0 + \gamma y_5 \\
& y_2 \le 0 + \gamma y_3 \\
& y_2 \le 0 + \gamma(0.5 y_4 + 0.5 y_5) \\
& y_1 \le 0 + \gamma y_2 \\
& y_1 \le 0 + \gamma(0.5 y_3 + 0.25 y_4 + 0.25 y_5) \\
& y_0 \le 0 + \gamma y_1 \\
& y_0 \le 0 + \gamma(0.5 y_2 + 0.25 y_3 + 0.125 y_4 + 0.125 y_5)
\end{aligned}$$



- y_i: expected overall cost if stating from State i.
- State 4 is a trap
- State 5 is the destination
- Each other state has two options: Go directly to the next state OR a short-cut go to other states with uncertainties

$$\mathbf{y^*_0 = y^*_1 = y^*_2 = y^*_3 = y^*_5 = 0}$$
$$\mathbf{y^*_4 = 1}$$

Physical Interpretation of the Maze-Run Dual

$$\max\ y_0 + y_1 + y_2 + y_3 + y_4 + y_5$$

$$\text{s.t.}\quad y_5 \le 0 + \gamma y_5 \quad (x_5)$$

$$y_4 \le 1 + \gamma y_5 \quad (x_4)$$

$$y_3 \le 0 + \gamma y_4 \quad (x_{3r})$$

$$y_3 \le 0 + \gamma y_5 \quad (x_{3b})$$

$$y_2 \le 0 + \gamma y_3 \quad (x_{2r})$$

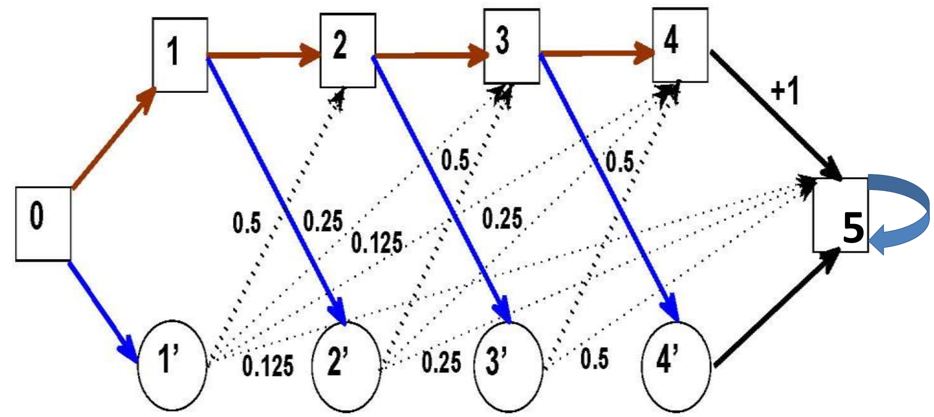
$$y_2 \le 0 + \gamma(0.5y_4 + 0.5y_5) \quad (x_{2b})$$

$$y_1 \le 0 + \gamma y_2 \quad (x_{1r})$$

$$y_1 \le 0 + \gamma(0.5y_3 + 0.25y_4 + 0.25y_5) \quad (x_{1b})$$

$$y_0 \le 0 + \gamma y_1 \quad (x_{0r})$$

$$y_0 \le 0 + \gamma(0.5y_2 + 0.25y_3 + 0.125y_4 + 0.125y_5) \quad (x_{0b})$$



x_j represents (discounted) how many expected times (frequency) actions j being taken in a policy.

The Dual of the Maze Example

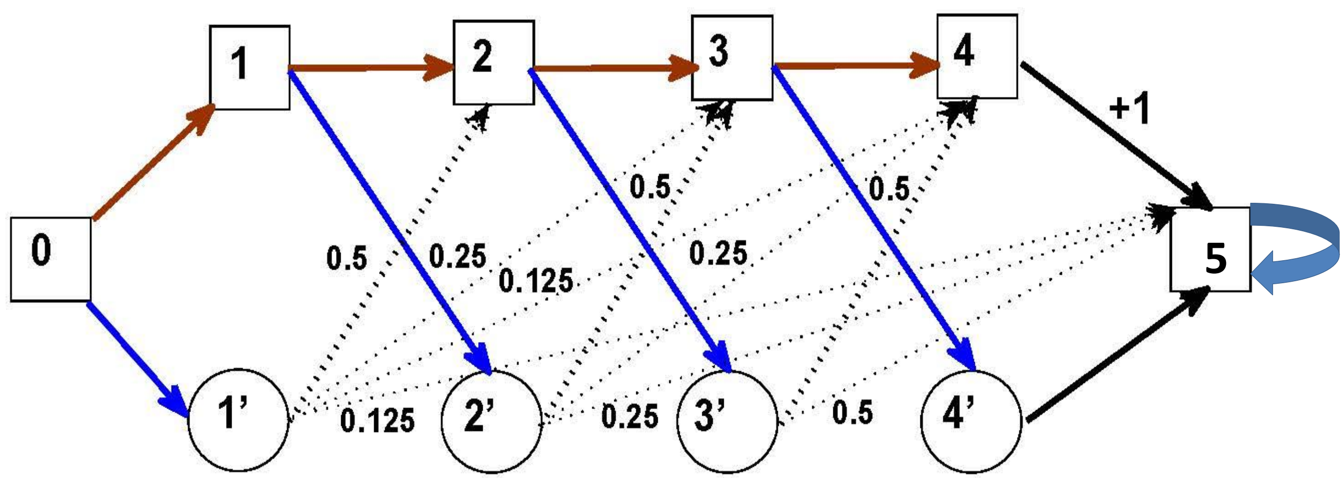
x:	$(0r)$	$(0b)$	$(1r)$	$(1b)$	$(2r)$	$(2b)$	$(3r)$	$(3b)$	(4)	(5)	b
c:	0	0	0	0	0	0	0	0	1	0	
(0)	1	1	0	0	0	0	0	0	0	0	1
(1)	$-\gamma$	0	1	1	0	0	0	0	0	0	1
(2)	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	0	0	1
(3)	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	1
(4)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	0	1
(5)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	0	$-\gamma$	$-\gamma$	$1-\gamma$	1

$$\begin{aligned} \min\ & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.}\ & A\boldsymbol{x} = \mathbf{e}, \quad (\boldsymbol{y}) \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

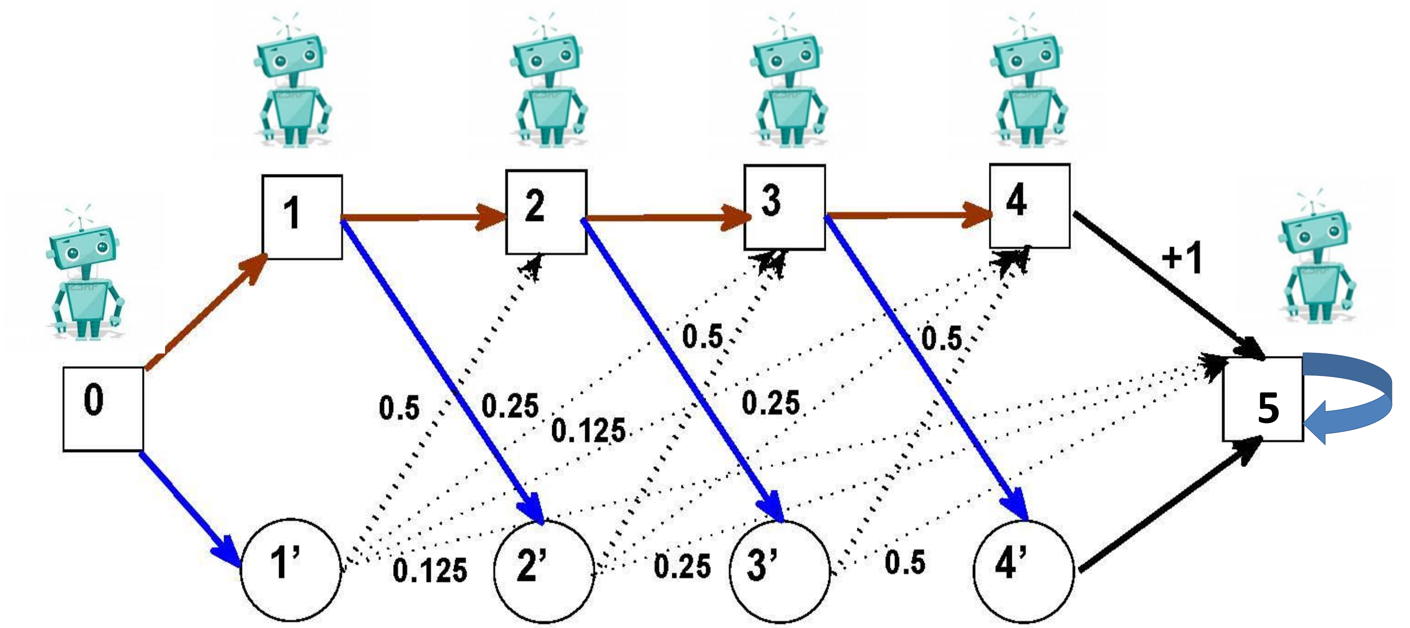
The optimal dual solution is

$$x^*_{0r} = 1,\ x^*_{1r} = 1+\gamma,\ x^*_{2r} = 1+\gamma+\gamma^2,\ x^*_{3b} = 1+\gamma+\gamma^2+\gamma^3,\ x^*_4 = 1,$$

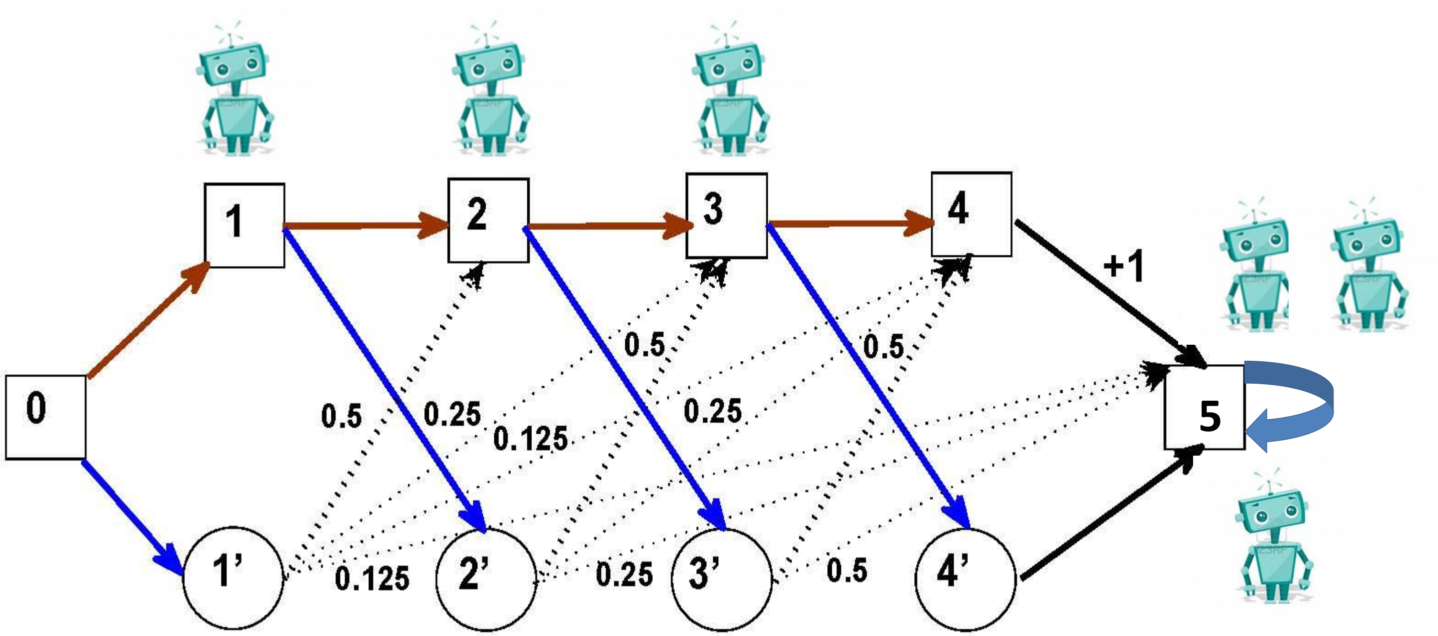
$$x^*_5 = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma}.$$

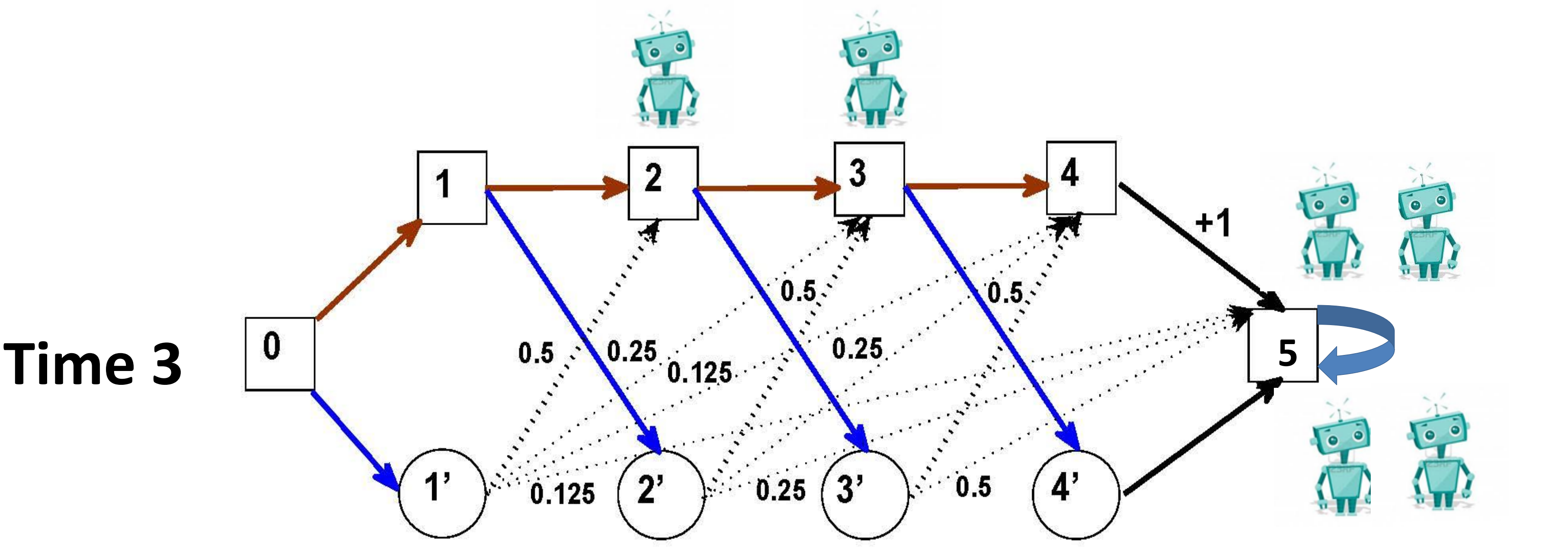


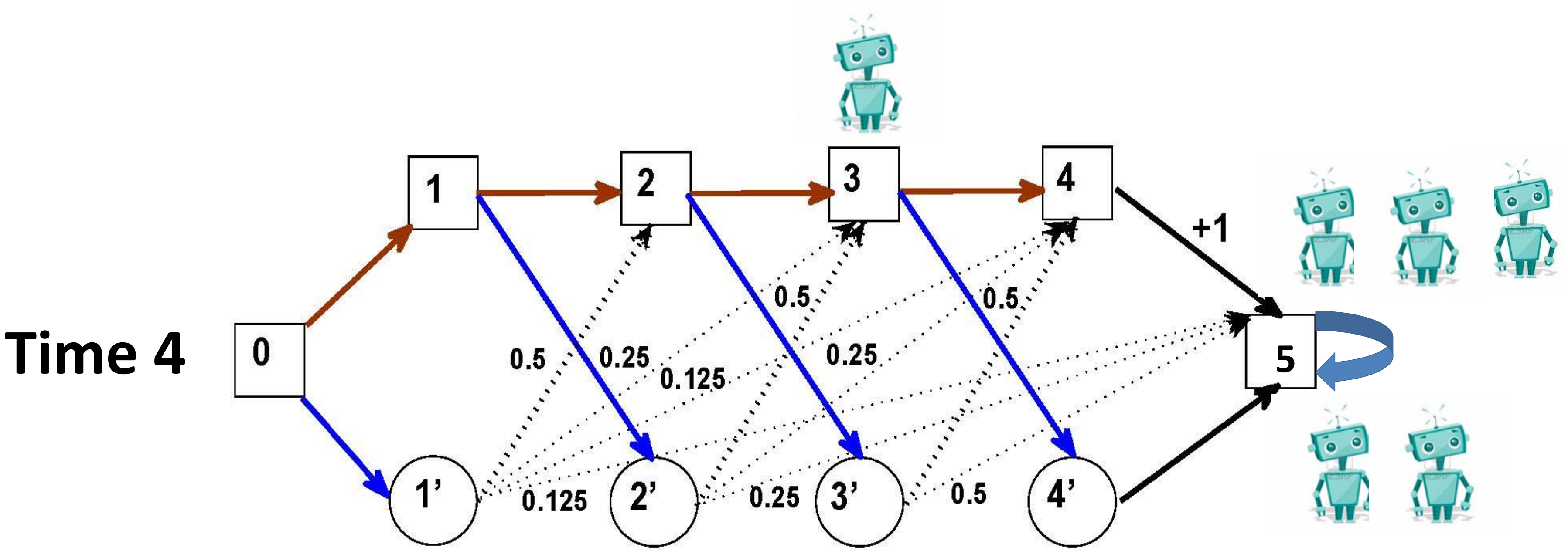
Time 1


0
1
2
3
4
5
1'
2'
3'
4'
+1
0.5
0.25
0.125
0.5
0.25
0.5
0.125
0.25
0.5

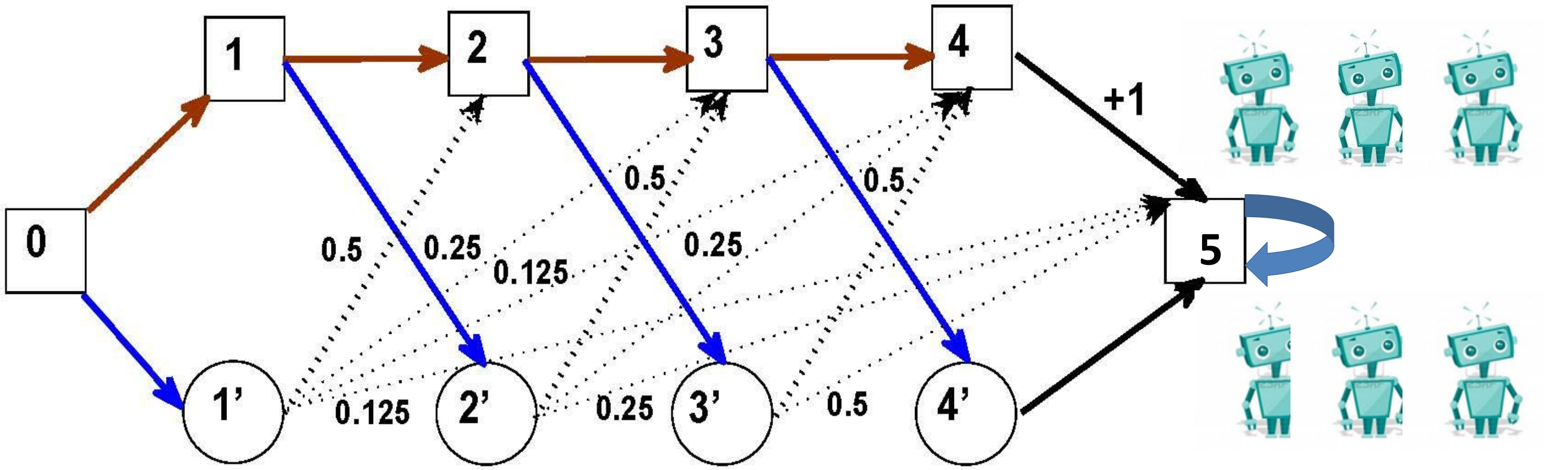
Time 2


0
1
2
3
4
5
1'
2'
3'
4'
+1
0.5
0.25
0.125
0.5
0.25
0.5
0.125
0.25
0.5


Time 3
0
1
2
3
4
5
1'
2'
3'
4'
+1
0.5
0.25
0.125
0.5
0.25
0.5
0.125
0.25
0.5
0.5
0.25


Time 4
0
1
2
3
4
5
1'
2'
3'
4'
+1
0.5
0.25
0.125
0.5
0.25
0.5
0.125
0.25
0.5
0.5
0.25

Time 5



Recall the optimal dual solution values are:

$$x_{0r}^* = 1,\ x_{1r}^* = 1+\gamma,\ x_{2r}^* = 1+\gamma+\gamma^2,\ x_{3b}^* = 1+\gamma+\gamma^2+\gamma^3,\ x_4^* = 1,$$

$$x_5^* = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma}.$$

Two-Person Zero-Sum Matrix Game

$$\begin{pmatrix} 3 & -1 & -3 \\ -3 & 1 & 4 \end{pmatrix} = P$$

P is the payoff matrix of a two-person, ”Column” and ”Row”, zero-sum game.
Player Column Player chooses column(s) to maximize the payoff to Column
Player Row chooses row(s) to minimize the payoff to Column

Pure Strategy: Each player chooses a single column (row).

Mixed or Randomized Strategy: Each player randomly chooses columns (rows) strategies with a fixed probability distribution.

Nash Equilibrium: No player can alter its probability distribution to achieve better expected payoff.

Two-Person Zero-Sum Matrix Game II

$$\begin{pmatrix} 3 & -1 & -3 \\ -3 & 1 & 4 \end{pmatrix}$$

Player Column Player: probabilities x_1 to choose column 1, x_2 to choose column 2, and x_3 to choose column 3. Then the expected payoff is

$3x_1 - x_2 - 3x_3$ if Player Row chooses row 1

$-3x_1 + x_2 + 4x_3$ if Player Row chooses row 2

Thus, Player Column would

$$\begin{array}{ll} \text{maximize}_{(x1,x2,x3)} & \min\{3x_1 - x_2 - 3x_3 , -3x_1 + x_2 + 4x_3\} \\ \text{s.t.} & X_1+x_2+x_3 = 1,\ (x_1,x_2,x_3) \geq 0 \end{array}$$

which can be cast as a linear program

$$\begin{array}{lll} \text{maximize}_{(x1,x2,x3,v)} & v & \\ \text{s.t.} & -3x_1 + x_2 + 3x_3 + v \leq 0 & y_1 \\ & 3x_1 - x_2 - 4x_3 + v \leq 0 & y_2 \\ & x_1 + x_2 + x_3 = 1, & u \\ & (x_1,x_2,x_3) \geq 0 & \end{array}$$

Two-Person Zero-Sum Matrix Game III

$$\begin{pmatrix} 3 & -1 & -3 \\ -3 & 1 & 4 \end{pmatrix}$$

Then, the dual of the linear program

$$\begin{array}{lll} \text{minimize}_{(y1,y2,u)} & u & \\ \text{s.t.} & u-(3y_1 - 3y_2) & \geq 0 \\ & u-(-y_1 + y_2) & \geq 0 \\ & u-(-3y_1 + 4y_2) & \geq 0 \\ & y_1 + y_2 & =1, \ (y_1,y_2) \geq 0 \end{array}$$

Interpretations:

Player Row: probabilities y_1 to choose row 1, y_2 to choose row 2. Then the expected payoff to Player Column is

$3y_1 - 3y_2$ if Player Column chooses column 1

$-y_1 + y_2$ if Player Column chooses column 2

$-3y_1+4y_2$ if Player Column chooses column 3;

and Player Row does

$$\begin{array}{ll} \text{minimize}_{(y1,y2)} & \max\{3y_1 - 3y_2, -y_1 + y_2, -3y_1+4y_2\} \\ \text{s.t.} & y_1+y_2 =1, \ (y_1,y_2) \geq 0 \end{array}$$

Robust Portfolio Management I

Two stocks with return rate 0.5 each, and stock 2 has more variance, and the two are negatively correlated

$$
\begin{array}{ll}
\min & (x_1)^2 + 2(x_2)^2 - 2x_1x_2 - 0.5x_1 - 0.5x_2 \\
\text{s.t.} & x_1 + x_2 = 1
\end{array}
$$

This is a convex optimization problem

FONC are sufficient: set the (partial) derivatives of LF

$$
\begin{aligned}
L(x_1, x_2, y) = (x_1)^2 + 2(x_2)^2 - 2x_1x_2 - 0.5x_1 - 0.5x_2 \\
- y(x_1 + x_2 - 1)
\end{aligned}
$$

to zeros

$$
\begin{aligned}
&2x_1 - 2x_2 - 0.5 - y = 0,\ \ 4x_2 - 2x_1 - 0.5 - y = 0 \\
&\Rightarrow x_2 = 0.5 + y \Rightarrow x_1 = 0.75 + 1.5y \Rightarrow \\
&\qquad \max \quad \phi(y) = -0.3125 - 0.25y - 1.25y^2
\end{aligned}
$$

Application: Robust Portfolio Management II

$$\begin{array}{ll} \min & (x_1)^2 + 2(x_2)^2 - 2x_1x_2 - \mu_1 x_1 - \mu_2 x_2 \\ \text{s.t.} & x_1 + x_2 = 1, \end{array}$$

But the return two return rates are uncertain, and they are in the range $\mu_1+\mu_2=1$, $(\mu_1)^2+(\mu_2)^2 \le 1$

$$\begin{array}{ll} \min & (x_1)^2 + 2(x_2)^2 - 2x_1x_2 + \\ & \left\{ \begin{array}{ll} \max_{\mu 1,\mu 2} - x_1\mu_1 - x_2\mu_2, & \\ \text{s.t.} & \mu_1 + \mu_2 = 1, (\mu_1)^2 + (\mu_2)^2 \le 1 \end{array} \right\} \\ \text{s.t.} & x_1 + x_2 = 1, \end{array}$$

The inner problem is maximization, representing the decision makers' complete risk reverse attitude

$$\begin{array}{ll} \min & (x_1)^2 + 2(x_2)^2 - 2x_1x_2 + \\ & \left\{ \begin{array}{ll} \min_{y1,y2} \dfrac{(-x_1 - y_1)^2 + (-x_2 - y_1)^2}{4y_2} + y_1 + y_2, & \\ \text{s.t.} & y_1 \text{ free}, y_2 \ge 0 \end{array} \right\} \\ \text{s.t.} & x_1 + x_2 = 1, \end{array}$$

Replacing the inner problem by its dual (see next slide).

The Dual of the Inner Problem

$$\begin{array}{lll} \max_{x1,x2} & c_1x_1 + c_2x_2 & \\ \text{s.t.} & x_1 + x_2 = 1, & \wedge\, y_1\text{: free} \\ & (x_1)^2 + (x_2)^2 \le 1, & \wedge\, y_2 \ge 0 \end{array}$$

Primal

$$L(x_1, x_2, y) = c_1x_1 + c_2x_2 - y_1(x_1 + x_2 - 1) \\ - y_2((x_1)^2 + (x_2)^2 - 1),$$

$$\begin{pmatrix} c_1 - y_1 - 2y_2x_1 \\ c_2 - y_1 - 2y_2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\phi(y) = \frac{(c_1 - y_1)^2 + (c_2 - y_1)^2}{4y_2} + y_1 + y_2,$$

$$\min\ \phi(y),\ \text{ s.t. }\ y_1 \text{ free}, y_2 \ge 0$$

Dual

Application: Robust Portfolio Management III

$$\begin{aligned}
\min\quad & (x_1)^2 + 2(x_2)^2 - 2x_1x_2 + \\
& \begin{Bmatrix} \min_{y1,y2} \dfrac{(-x_1-y_1)^2+(-x_2-y_1)^2}{4y_2} + y_1 + y_2, \\ \text{s.t.} \qquad y_1 \text{ free}, y_2 \ge 0 \end{Bmatrix} \\
\text{s.t.}\quad & x_1 + x_2 = 1,
\end{aligned}$$

The objectives of the outer and inner problems are now aligned, so that we can combine them into a joint single layer problem

$$\begin{aligned}
&\min\ (x_1)^2 + 2(x_2)^2 - 2x_1x_2 + \frac{(x_1+y_1)^2+(x_2+y_1)^2}{4y_2} + y_1 + y_2 \\
&\text{s.t.}\ \ x_1 + x_2 = 1, y_1 \text{ free}, y_2 \ge 0
\end{aligned}$$

$$\begin{aligned}
&\min\ (x_1)^2 + 2(x_2)^2 - 2x_1x_2 + \sqrt{(x_1+y_1)^2+(x_2+y_1)^2} + y_1 \\
&\text{s.t.}\ \ x_1 + x_2 = 1, y_1 \text{ free}
\end{aligned}$$

Now, given x_1 and x_2, how to find minimizer y_1?

$$\begin{aligned}
&\min\quad y_1 + r \\
&\text{s.t.} \sqrt{(x_1+y_1)^2+(x_2+y_1)^2} \le r, y_1 \text{free},
\end{aligned}$$

$$= -\min\{x_1, x_2\}$$

Application: Robust Portfolio Management IV

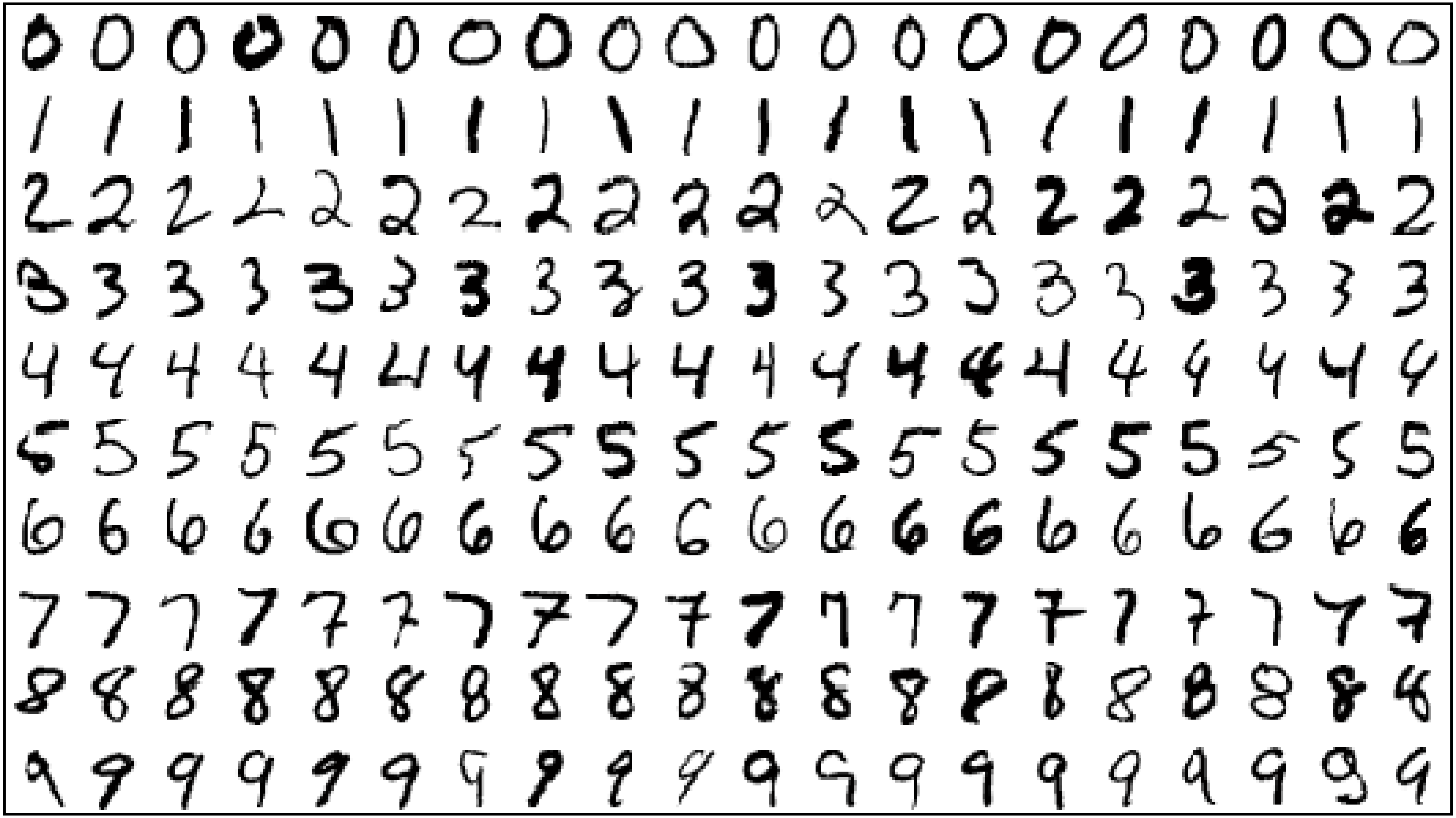
$$\boxed{\begin{array}{l} \min\ (x_1)^2 + 2(x_2)^2 - 2x_1x_2 - \min\{x_{1,} x_2\} \\ \text{s.t.}\ \ x_1 + x_2 = 1. \end{array}}$$

$$\boxed{\begin{array}{l} \min\ (x_1)^2 + 2(x_2)^2 - 2x_1x_2 - z \\ \text{s.t.}\ \ x_1 + x_2 = 1,\ \ x_1 \geq z,\ \ x_2 \geq z \end{array}}$$

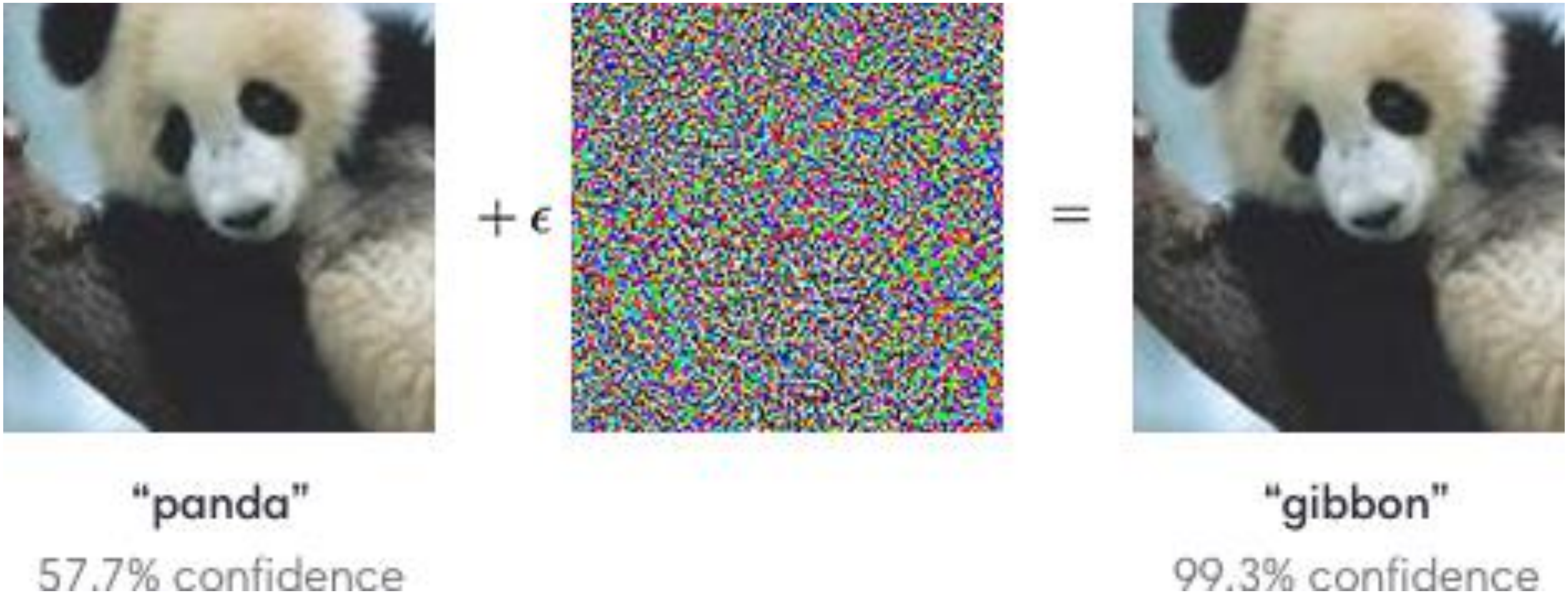
This is a convex optimization problem

$$\boxed{\begin{array}{l} \text{FONC are sufficient : try } z = x_2 \\ 2x_1 - 2x_2 - y = 0,\ 4x_2 - 2x_1 - 1 - y = 0 \\ \Rightarrow 6x_2 - 4x_1 - 1 = 0 \Rightarrow x_2 = \dfrac{1}{2} \Rightarrow x_1 = \dfrac{1}{2} \end{array}}$$

- Deep Learning Based on Sample Writings



Sample-Based Learning May be Vulnerable



[Goodfellow et al. 14]

The Dual of the Information Market Problem

The ith order is given as triple ($\mathbf{a}_i \in R^m, \pi_i \in R_+, q_i \in R_+$):

$$\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{im})$$

is the betting indication row vector where each component is either 1 or 0, where 1 is winning state and 0 is non-winning state;

π_i is the bidding price for one share of such a contract, and

q_i is the maximum number of shares the bidder like to own.

A contract /share on an order is a paper agreement so that on maturity it is worth a notional \$1 dollar if the order includes the winning state and worth \$0 otherwise.

Let x_i be the number of units awarded to the ith order.

A Risk-Free Mechanism of Market Maker

Corresponding Dual Variables

$$
\begin{array}{llll}
\max & \pi^T \mathbf{x} - x_{n+1} & & \\
\text{s.t.} & A^T \mathbf{x} - \mathbf{1} \cdot x_{n+1} \leq \mathbf{0} & & \boldsymbol{p} \\
& \mathbf{x} \leq \mathbf{q} & & \boldsymbol{s} \\
& \mathbf{x} \geq \mathbf{0} & & \\
& x_{n+1} \text{ free} & &
\end{array}
$$

where $\mathbf{1}$ is the vector of all ones.

$\pi^T \mathbf{x}$: the revenue amount can be collected.

x_{n+1}: the worst-case cost (amount need to pay to the winners).

The Dual: Regression with "Under-Bid" Filtering

$$\begin{array}{lll} \min & \boldsymbol{q}^T \boldsymbol{s} & \\ \text{s.t.} & A\boldsymbol{p} + \boldsymbol{s} & \geq \boldsymbol{\pi}, \\ & -\boldsymbol{1}^T \boldsymbol{p} & = -1, \\ & (\boldsymbol{p}, \boldsymbol{s}) & \geq 0. \end{array}$$

$\boldsymbol{p}_j$: the shadow/dual price of state j;

$\boldsymbol{a}_i\boldsymbol{p}$: the ith order unit cost at prices $\boldsymbol{p}$;

$\boldsymbol{s}_j$: the unit profit from the jth order ($\boldsymbol{s}=\max\{\boldsymbol{0}, \boldsymbol{\pi}-A\boldsymbol{p}\}$)

The dual problem is to minimize the total "Regression Loss" collected from the (competitive or high-bid) orders, $\boldsymbol{q}^T\boldsymbol{s}$.

ReLu-Regression for Probability Distribution/Information

$$\begin{array}{lll} \min & \boldsymbol{q}^T \max\{\boldsymbol{0}, \boldsymbol{\pi}\text{-}A\boldsymbol{p}\} & \\ \text{s.t.} & \boldsymbol{1}^T \boldsymbol{p} & = 1, \\ & \boldsymbol{p} & \geq 0 \end{array}$$

$\boldsymbol{p}_j$: the shadow-price/probability estimation of state j;

$\boldsymbol{a}_i\boldsymbol{p}$: the ith order unit cost at prices $\boldsymbol{p}$;

$\boldsymbol{\pi}_i$: the ith order bidding price;

$\boldsymbol{q}_i$: the ith order quantity limit;

The dual problem is a (unsupervised) SVM problem to identify the "HIGH" (competitive) bids from "LOW" (noncompetitive) bids, where $(1;\boldsymbol{p})$ is the gradient vector of the separating hyperplane where the intercept is zero.

The World Cup Betting Example

Orders Filled

Order	Price Limit	Quantity Limit	Filled	Argentina	Brazil	Italy	Germany	France
1	0.75	10	5	1	1	1		
2	0.35	5	5				1	
3	0.40	10	5	1		1		1
4	0.95	10	0	1	1	1	1	
5	0.75	5	5		1		1	

State Prices

	Argentina	Brazil	Italy	Germany	France
Price	0.20	0.35	0.20	0.25	0.00

Online Retail Sell

- There is a fixed selling period or number of buyers; and there is a fixed inventory of goods
- Customers come and require a bundle of goods and make a bid
- Decision: To sell or not to sell to each individual customer?
- Objective: Maximize the revenue.

Bid #	$100	$30	….	…	…	Inventory
Decision	x1	x2				
Pants	1	0	….	…	…	100
Shoes	1	0				50
T-Shirts	0	1				500
Jackets	0	0				200
Hats	1	1	…	…	…	1000

On-Line Retailer Linear Programming

- Off-line Problem is an (0,1) linear program that can be relaxed as LP
- But now trader/Bidders come one by one sequentially,
- The retailer has to make the decision as soon as an order arrives with the arrived combinatorial order/bid $(\mathbf{a}_k, \pi_k)$
- The retailer faces a dilemma:
  - **To sell or not to sell – this is the decision**
- Optimal Policy or Online Algorithm?

$$
\begin{array}{lll}
\max & \sum_{j=1}^{n} \pi_j x_j & \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & \forall i = 1, \dots, m \\
& x_j = \{0 \; or \; 1\} & \forall j = 1, \dots, n
\end{array}
$$

$$
\begin{array}{lll}
\max & \sum_{j=1}^{n} \pi_j x_j & \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & \forall i = 1, \dots, m \\
& 0 \le x_j \le 1 & \forall j = 1, \dots, n
\end{array}
$$

Off-Line LP Relaxation

CSC of Off-Line Retailer Linear Programming

- Let the optimal solution be $\boldsymbol{x}^*$ and the optimal shadow piece be $\boldsymbol{y}^*$
- Then from the CSC conditions:

$$x_j^* = 1 \text{ if } \pi_j > \mathbf{a}_j^T \boldsymbol{y}^*$$

$$x_j^* = 0 \text{ if } \pi_j < \mathbf{a}_j^T \boldsymbol{y}^*$$

$$x_j^* = \text{fraction if } \pi_j = \mathbf{a}_j^T \boldsymbol{y}^*$$

- If we know $\boldsymbol{y}^*$, the online decision would be easy!

$$\begin{array}{lll} \max & \sum_{j=1}^{n} \pi_j x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & \forall i = 1, \ldots, m \\ & 0 \le x_j \le 1 & \forall j = 1, \ldots, n \end{array}$$

Off-Line LP Relaxation

Online Algorithm and Price-Mechanism

- **Learn "ideal" itemized optimal prices**
- **Use the prices to price each bid**
- **Accept if it is a over bid, and reject otherwise**

Bid #	$100	$30	….	…	…	Inventory	Price?
Decision	x1	x2					
Pants	1	0	….	…	…	100	45
Shoes	1	0				50	45
T-Shirts	0	1				500	10
Jackets	0	0				200	55
Hats	1	1	…	…	…	1000	15

Such ideal prices exist, and they are shadow/dual prices of the offline LP

How to Learn the Shadow Prices Sequentially?

- Sequential Linear Programming Mechanism (SLPM)
  - Solving the LP based on immediately past several periods' data and use the resulted optimal shadow prices to make decision for the next period orders; and repeat when the current period is over.

- The shadow prices are updated periodically and being used to make online decisions for the next period.