Fixed Income and Risk Management

Interest Rate Models

Fall 2003, Term 2

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Agenda and key issues

• Pricing with binomial trees
  – Replication
  – Risk-neutral pricing
• Interest rate models
  – Definitions
  – Uses
  – Features
  – Implementation
• Binomial tree example
• Embedded options
  – Callable bond
  – Putable bond
• Factor models
  – Spot rate process
  – Drift and volatility functions
  – Calibration
1-period binomial model

- Stock with price $S = 60$ and one-period risk-free rate of $r = 20\%$
- Over next period stock price either falls to $30$ or rises to $90$
  \[
  S_d = 30, \quad S_u = 90
  \]
- Call option with strike price $K = 60$ pays either $0$ or $30$
  \[
  C_d = 0, \quad C_u = 30
  \]
- Buy $\Delta = \frac{1}{2}$ share of stock and borrow $L = 12.50$
  \[
  S_{u/2} - 1.2 \times 12.5 = 45 - 15 = 30
  \]
  \[
  S_{d/2} - 1.2 \times 12.5 = 15 - 15 = 0
  \]
1-period binomial model (cont)

- Portfolio replicates option payoff \( \Rightarrow C = $17.50 \)
- Solving for replicating portfolio
  - Buy \( \Delta \) shares of stock and borrow \( L \)
  - If stock price rises to $90, we want the portfolio to be worth
    \[
    90 \times \Delta - 1.2 \times L = $30
    \]
  - If stock price drops to $30, we want the portfolio to be worth
    \[
    30 \times \Delta - 1.2 \times L = $0
    \]
  - \( \Delta = 0.5 \) and \( L = $12.50 \) solve these two equations
1-period binomial model (cont)

• “Delta”
  – $\Delta$ is chosen so that the value of the replicating portfolio ($\Delta \times S - L$) has the same sensitivity to $S$ as the option price $C$
    $$\Delta = \frac{dC}{dS} = \frac{$30 - $0}{$90 - $30} = \frac{1}{2}$$
  – $\Delta$ is called the hedge ratio of “delta” of the option
  – Delta-hedging an option is analogous to duration-hedging a bond
1-period binomial model (cont)

• Very important result
  
The option price does not depend on the probabilities of a stock price up-move or down-move

• Intuition
  – If $C \neq \Delta \times S - L$, there exist an arbitrage opportunity
  – Arbitrage opportunities deliver riskless profits
  – Riskless profits cannot depend on probabilities
  – Therefore, the option price cannot depend on probabilities
1-period binomial model (cont)

- Unfortunately, this simple replication argument does not work with 3 or more payoff states

\[
\begin{align*}
S & \quad S_u \\
S_m & \quad C = ? \\
S_d & \quad C_u \\
S_{du} & \quad C_m \\
S_{dd} & \quad C_d
\end{align*}
\]

- Rather than increase the number of payoff states per period, increase the number of binomial periods ⇒ binomial tree

Recombining

Non-recombining
1-period binomial model (cont)

• Define
  – \( u = 1 + \) return if stock price goes up
  – \( d = 1 + \) return if stock price goes down
  – \( r \) = per-period riskless rate (constant for now)
  – \( p \) = probability of stock price up-move

• No arbitrage requires \( d \leq 1 + r \leq u \)

• Stock and option payoffs

\[
\begin{align*}
S \quad & S \times u \\
& S \times d \\
C \quad & C_u = f(S \times u) \\
& C_d = f(S \times d)
\end{align*}
\]
1-period binomial model (cont)

- Payoff of portfolio of $\Delta$ shares and $L$ dollars of borrowing
  \[ \Delta \times S \times u - L \times (1+r) \]
  \[ \Delta \times S \times d - L \times (1+r) \]

- Replication requires
  \[ \Delta \times S \times u - L \times (1+r) = C_u \]
  \[ \Delta \times S \times d - L \times (1+r) = C_d \]

- Two equations in two unknowns ($\Delta$ and $L$) with solution
  \[ \Delta = \frac{C_u - C_d}{S \times (u - d)} \]
  \[ L = \frac{d \times C_u - u \times C_d}{(1+r) \times (r-d)} \]

- Option price
  \[ C = \Delta \times S - L \]
Risk-neutral pricing (cont)

• Define

\[ q = \frac{(1 + r) - d}{u - d} \quad (1 - q) = \frac{u - (1 + r)}{u - d} \]

• No-arbitrage condition \( d \leq 1 + r \leq u \) implied \( 0 \leq q \leq 1 \)

• Rearrange option price

\[
C = \Delta \times S - L \\
= \frac{C_u - C_d}{S \times (u - d)} \times S - \frac{d \times C_u - u \times C_d}{(1 + r) \times (u - d)} \\
\ldots \\
= \frac{q \times C_u + (1 - q) \times C_d}{1 + r}
\]
Risk-neutral pricing (cont)

- Interpretation of $q$
  - Expected return on the stock
    \[
    E \left[ \frac{S_1}{S_0} \right] = \frac{p \times S \times u + (1-p) \times S \times d}{S} = p \times u + (1-p) \times d
    \]
  - Suppose we were risk-neutral
    \[
    E \left[ \frac{S_1}{S_0} \right] = p \times u + (1-p) \times d = (1 + r)
    \]
  - Solving for $p$
    \[
    p = \frac{(1 + r) - d}{u - d} = q
    \]

- Very, very important result
  $q$ is the probability which sets the expected return on the stock equal to the riskfree rate $\Rightarrow$ risk-neutral probability
Risk-neutral pricing (cont)

• Very, very, very important result

  The option price equals its expected payoff discounted by the riskfree rate, where the expectation is formed using risk-neutral probabilities instead of real probabilities ⇒ risk-neutral pricing

• Risk-neutral pricing extends to multiperiod binomial trees and applies to all derivatives which can be replicated

  Derivatives price = \( PV_r \left[ E^q [\text{payoff}] \right] \)
Risk-neutral pricing intuition

• Step 1
  – Derivatives are priced by no-arbitrage
  – No-arbitrage does not depend on risk preferences or probabilities

• Step 2
  – Imagine a world in which all security prices are the same as in the real world but everyone is risk-neutral (a “risk-neutral world”)
  – The expected return on any security equals the risk-free rate $r$

• Step 3
  – In the risk-neutral world, every security is priced as its expected payoff discounted by the risk-free rate, including derivatives
  – Expectations are taken wrt the risk-neutral probabilities $q$

• Step 4
  – Derivative prices must be the same in the risk-neutral and real worlds because there is only one no-arbitrage price
2-period binomial model

- Stock and option payoffs

\[
\begin{align*}
S & \quad S \times u^2 \\
S \times u & \quad S \times u \times d \\
S \times d & \quad C = ? \\
S \times d^2 & \quad C = ? \\
\end{align*}
\]

- By risk-neutral pricing

\[
C = \frac{q^2 \times C_{uu} + 2 \times q \times (1 - q) \times C_{ud} + (1 - q)^2 \times C_{dd}}{(1 + r)^2}
\]
3-period binomial model

• Stock and option payoffs

\[
\begin{align*}
S & \quad S \times u \times d \\
S \times d & \quad S \times u \times d^2 \\
S \times u & \quad S \times u^2 \times d \\
S \times u^2 & \quad S \times u^3 \\
S \times d^2 & \quad S \times d^3 \\
S & \quad S
\end{align*}
\]

\[
C \quad C_{uu} = f(S \times u^3) \\
C_{uu} = f(S \times u^2 \times d) \\
C_{udd} = f(S \times u \times d^2) \\
C_{ddd} = f(S \times d^3)
\]

• By risk-neutral pricing

\[
C = \frac{1}{(1 + r)^3} \times \left[ q^3 \times C_{uuu} + 3 \times q^2 \times (1 - q) \times C_{uud} + \ldots + 3 \times q \times (1 - q)^2 \times C_{udd} + (1 - q)^3 \times C_{ddd} \right]
\]
Definitions

• An interest rate model describes the dynamics of either
  – 1-period spot rate
  – Instantaneous spot rate = $t$-year spot rate $r(t)$ as $t \to 0$

• Variation in spot rates is generated by either
  – One source of risk ⇒ single-factor models
  – Two or more sources of risk ⇒ multifactor models
Model uses

• Characterize term structure of spot rates to price bonds

• Price interest rate and bond derivatives
  – Exchange traded (e.g., Treasury bond or Eurodollar options)
  – OTC (e.g., caps, floors, collars, swaps, swaptions, exotics)

• Price fixed income securities with embedded options
  – Callable or putable bonds

• Compute price sensitivities to underlying risk factor(s)

• Describe risk-reward trade-off
Model features

• Interest rate models should be
  – Arbitrage free = model prices agree with current market prices
    ▪ Spot rate curve
    ▪ Coupon yield curve
    ▪ Interest rate and bond derivatives
  – Time-consistent = model implied behavior of spot rates and bond prices agree with their observed behavior
    ▪ Mean reversion
    ▪ Conditional heteroskedasticity
    ▪ Term structure of volatility and correlation structure

• Developing an interest rate model which is both arbitrage free and time consistent is the holy grail of fixed income research
Model implementation

• In practice, two model implementations
  – Cross-sectional calibration
    ▪ Calibrate model to match exactly all market prices of liquid securities on a single day
    ▪ Used for pricing less liquid securities and derivatives
    ▪ Arbitrage free but probably not time-consistent
    ▪ Usually one or two factors
  – Time-series estimation
    ▪ Estimate model using a long time-series of spot rates
    ▪ Used for hedging and asset allocation
    ▪ Time-consistent but not arbitrage free
    ▪ Usually two and more factors
Spot rate tree

- 1-period spot rates ($m$-period compounded APR)

\[ r_{0,0}(1) = 10\% \]
\[ r_{1,0}(1) = 9\% \]
\[ r_{1,1}(1) = 11\% \]
\[ r_{2,0}(1) = 8\% \]
\[ r_{2,1}(1) = 10\% \]
\[ r_{2,2}(1) = 12\% \]

- Notation
  - \( r_{ij}(n) = n\)-period spot rate \( i \) periods in the future after \( j \) up-moves
  - \( \Delta t \) = length of a binomial step in units of years

- Set \( \Delta t = \frac{1}{m} \) and \( m = 2 \)
- Assume \( q_{ij} = 0.5 \) for all steps \( i \) and nodes \( j \)
**Road-map**

• **Calculate step-by-step**
  – Implied spot rate curve $r_{0,0}^{(1)}$, $r_{0,0}^{(2)}$, $r_{0,0}^{(3)}$
  – Implied changes in the spot rate curve

\[
\begin{array}{c}
  r_{0,0}^{(1)}, r_{0,0}^{(2)} \\
  \downarrow \quad r_{1,0}^{(1)}, r_{1,0}^{(2)} \\
  \downarrow \quad r_{1,1}^{(1)}, r_{1,1}^{(2)}
\end{array}
\]

– Price 8% 1.5-yr coupon bond
– Price 1-yr European call option on 8% 1.5-yr coupon bond
– Price 1-yr American put option on 8% 1.5-yr coupon bond
1-period zero-coupon bond prices

• At time 0

\[ P_{0,0}(1) = ? \]

\[ P_{1,1}(0) = $100 \]
\[ P_{1,0}(0) = $100 \]

\[
P_{0,0}(1) = \frac{q_{0,0} \times P_{1,1}(0) + (1 - q_{0,0}) \times P_{1,0}(0)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}}
\]

\[
= \frac{$100}{(1 + 0.1/2)^1} = $95.24
\]
1-period zero-coupon bond prices (cont)

• At time 1

\[
P_{1,1}(1) = \frac{q_{1,1} \times P_{2,2}(0) + (1 - q_{1,1}) \times P_{2,1}(0)}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}}
\]

\[
= \frac{$100}{(1 + 0.11/2)^1} = $94.79
\]

\[
P_{1,0}(1) = \frac{q_{1,0} \times P_{2,1}(0) + (1 - q_{1,0}) \times P_{2,0}(0)}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}}
\]

\[
= \frac{$100}{(1 + 0.09/2)}^1 = $95.69
\]
1-period zero-coupon bond prices (cont)

• At time 2

\[ P_{2,2}(1) = \frac{q_{2,2} \times P_{3,3}(0) + (1 - q_{2,2}) \times P_{3,2}(0)}{(1 + r_{2,2}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{\$100}{(1 + 0.12/2)^1} = \$94.34 \]

\[ P_{2,1}(1) = \frac{q_{2,1} \times P_{3,2}(0) + (1 - q_{2,1}) \times P_{3,1}(0)}{(1 + r_{2,1}(1)/m)^{1 \times \Delta t \times m}} = \$95.24 \]

\[ P_{2,0}(1) = \frac{q_{2,0} \times P_{3,1}(0) + (1 - q_{2,0}) \times P_{3,0}(0)}{(1 + r_{2,0}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{\$100}{(1 + 0.08/2)^1} = \$96.15 \]
1-period zero-coupon bond prices (cont)

\[
\begin{align*}
P_{0,0}(1) &= $95.24 \\
P_{1,0}(1) &= $95.69 \\
P_{1,1}(1) &= $94.79 \\
P_{2,0}(1) &= $96.15 \\
P_{2,1}(1) &= $95.25 \\
P_{2,2}(1) &= $94.34
\end{align*}
\]
2-period zero-coupon bond prices

• At time 0

\[ P_{0,0}(2) = \frac{q_{0,0} \times P_{1,1}(1) + (1 - q_{0,0}) \times P_{1,0}(1)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{\frac{1}{2} \times $94.79 + \frac{1}{2} \times $95.69}{(1 + 0.1/2)^1} = $90.71 \]

• Implied 2-period spot rate

\[ P_{0,0}(2) = \frac{$100}{(1 + r_{0,0}(2)/m)^{2 \times \Delta t \times m}} \Rightarrow r_{0,0}(2) = 9.9976\% \]
2-period zero-coupon bond prices (cont)

- At time 1

\[
P_{1,1}(2) = \frac{q_{1,1} \times P_{2,2}(1) + (1 - q_{1,1}) \times P_{2,1}(1)}{(1 + r_{1,1}(1)/m)^{1\times\Delta t\times m}}
\]

\[
= \frac{\frac{1}{2} \times $94.34 + \frac{1}{2} \times $95.24}{(1 + 0.11/2)^1} = $89.85
\]

\(\Rightarrow r_{1,1}(2) = 10.9976\%
\)

\[
P_{1,0}(2) = \frac{q_{1,0} \times P_{2,1}(1) + (1 - q_{1,0}) \times P_{2,0}(1)}{(1 + r_{1,0}(1)/m)^{1\times\Delta t\times m}}
\]

\[
= \frac{\frac{1}{2} \times $95.24 + \frac{1}{2} \times $96.15}{(1 + 0.09/2)^1} = $91.58
\]

\(\Rightarrow r_{1,0}(2) = 8.9976\%\)
3-period zero-coupon bond price

- At time 0

\[ P_{0,0}(3) = ? \]

\[ \begin{align*}
P_{1,1}(2) &= $89.85 \\
P_{1,0}(2) &= $91.58 \\
\end{align*} \]

\[ P_{0,0}(3) = \frac{q_{0,0} \times P_{1,1}(2) + (1 - q_{0,0}) \times P_{1,0}(2)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} = \frac{\frac{1}{2} \times $89.85 + \frac{1}{2} \times $91.58}{(1 + 0.1/2)^1} = $86.39 \]

- Implied 3-period spot rate

\[ P_{0,0}(3) = \frac{$100}{(1 + r_{0,0}(3)/m)^{3 \times \Delta t \times m}} \Rightarrow r_{0,0}(3) = 9.9937\% \]
Implied spot rate curve

• Current spot rate curve is slightly downward sloping

\[ r_{0,0}(1) = 10.0000\% \]
\[ r_{0,0}(2) = 9.9976\% \]
\[ r_{0,0}(3) = 9.9937\% \]

• From one period to the next, the spot rate curve shifts in parallel

\[ r_{1,1}(1) = 11.0000\% \]
\[ r_{1,1}(2) = 10.9976\% \]
\[ r_{1,0}(1) = 9.0000\% \]
\[ r_{1,0}(2) = 8.9976\% \]
**Coupon bond price**

- 8% 1.5-year (3-period) coupon bond with cashflow

```
   $0.00
  /     \\  \
 $4.00  $4.00
  |     |   |   |
$4.00 $4.00 $4.00 $104.00
```

$104.00

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Interest Rate Models
Coupon bond price (cont)

- Discounting terminal payoffs by 1 period

\[
P_{1,1} = ? \\
P_{1,0} = ?
\]

\[
P_{2,2} = \frac{104.00}{1.06} = 98.11
\]

\[
P_{2,1} = \frac{104.00}{1.05} = 99.05
\]

\[
P_{2,0} = \frac{104.00}{1.04} = 100
\]

\[
P_{3,0} = 104.00
\]

\[
P_{3,1} = 104.00
\]

\[
P_{3,2} = 104.00
\]

\[
P_{3,3} = 104.00
\]
Coupon bond price (cont)

• By risk-neutral pricing

\[ P_{0,0} = ? \]

\[ P_{1,1} = \frac{c + q_{1,1} \times P_{2,2} + (1 - q_{1,1}) \times P_{2,1}}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{4.00 + \frac{1}{2} \times 98.11 + \frac{1}{2} \times 99.05}{1.055} \]

\[ = \text{?} \]

\[ P_{1,0} = ? \]

\[ P_{2,0} = \frac{104.00}{1.04} = 100 \]

\[ P_{2,1} = \frac{104.00}{1.05} = 99.05 \]

\[ P_{2,2} = \frac{104.00}{1.06} = 98.11 \]

\[ P_{3,0} = 104.00 \]

\[ P_{3,1} = 104.00 \]

\[ P_{3,2} = 104.00 \]

\[ P_{3,3} = 104.00 \]
**Coupon bond price (cont)**

- By risk-neutral pricing (cont)

\[ P_{1,0} = \frac{c + q_{1,0} \times P_{2,1} + (1 - q_{1,0}) \times P_{2,0}}{(1 + r_{1,0}(1)/m)^{\Delta t \times m}} \]

\[ = \frac{4.00 + \frac{1}{2} \times 99.05 + \frac{1}{2} \times 100.00}{1.045} \]

\[ = \frac{97.23}{1.05} = 92.30 \]

\[ P_{1,1} = \frac{104.00}{1.05} = 99.05 \]

\[ P_{1,2} = \frac{104.00}{1.06} = 98.08 \]

\[ P_{2,2} = \frac{104.00}{1.06} = 98.11 \]

\[ P_{2,1} = \frac{104.00}{1.05} = 99.05 \]

\[ P_{2,0} = \frac{104.00}{1.04} = 100.00 \]

\[ P_{3,0} = 104.00 \]

\[ P_{3,1} = 104.00 \]

\[ P_{3,2} = 104.00 \]

\[ P_{3,3} = 104.00 \]
Binomial tree example

**Coupon bond price (cont)**

- By risk-neutral pricing (cont)

\[
P_0,0 = \frac{c + q_{0,0} \times P_{1,1} + (1 - q_{0,0}) \times P_{1,0}}{(1 + r_{0,0}(1)/m)^{1\times\Delta t\times m}}
\]

\[
P_0,0 = \frac{4.00 + \frac{1}{2} \times 97.23 + \frac{1}{2} \times 99.07}{1.05}
\]

\[
P_0,0 = \frac{4.00 + 48.615 + 49.535}{1.05}
\]

\[
P_0,0 = \frac{102.15}{1.05}
\]

\[
P_0,0 = 97.28
\]
European call on coupon bond

- 1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays $\max[0, P_{T} - K]$

\[
\begin{align*}
V_{0,0} &= ? \\
V_{1,0} &= ? \\
V_{1,1} &= ? \\
V_{2,0} &= \max[0, 100.00 - 99.00] = 1.00 \\
V_{2,1} &= \max[0, 99.05 - 99.00] = 0.05 \\
V_{2,2} &= \max[0, 98.11 - 99.00] = 0
\end{align*}
\]
European call on coupon bond

- 1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K = $99.00 pays max[ 0, $P_2 - K$ ]

\[ V_{0,0} = ? \]
\[ V_{1,1} = $0.0237 \]
\[ V_{1,0} = ? \]
\[ V_{2,2} = \max[0, $98.11 - $99.00$] = 0 \]
\[ V_{2,1} = \max[0, $99.05 - $99.00$] = $0.05 \]
\[ V_{2,0} = \max[0, $100.00 - $99.00$] = $1.00 \]

\[ V_{1,1} = \frac{q_{1,1} \times V_{2,2} + (1 - q_{1,1}) \times V_{2,1}}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}} \]
\[ = \frac{0.5 \times $0.00 + 0.5 \times $0.05}{1.055} \]
European call on coupon bond

• 1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K = $99.00 pays max[ 0, $P_2,? – K ]

\[ V_{1,1} = \frac{q_{1,0} \times V_{2,1} + (1 - q_{1,0}) \times V_{2,0}}{(1 + r_{1,0}/m)^{1 \times \Delta t \times m}} \]

\[ V_{1,0} = \frac{0.5 \times $0.05 + 0.5 \times $1.00}{1.045} \]

\[ V_{1,0} = \frac{0.5 \times $0.05 + 0.5 \times $1.00}{1.045} \]

\[ V_{2,2} = \text{max}[0, 98.11 - 99.00] \]

\[ = 0 \]

\[ V_{2,1} = \text{max}[0, 99.05 - 99.00] \]

\[ = $0.05 \]

\[ V_{2,0} = \text{max}[0, 100.00 - 99.00] \]

\[ = $1.00 \]
European call on coupon bond

- 1-yr European style call option on 8% 1.5-yr coupon bond with strike price \( K = $99.00 \) pays \( \max[0, P_{2,?} - K] \)

\[
V_{0,0} = $0.2505
\]

\[
V_{1,0} = $0.5024
\]

\[
V_{1,1} = $0.0237
\]

\[
V_{2,0} = \max[0, $100.00 - $99.00] = $1.00
\]

\[
V_{2,1} = \max[0, $99.05 - $99.00] = $0.05
\]

\[
V_{2,2} = \max[0, $98.11 - $99.00] = 0
\]

\[
V_{0,0} = \frac{q_{0,0} \times V_{1,1} + (1 - q_{0,0}) \times V_{1,0}}{(1 + r_{0,0}(1)/m)^{1 \times 1 \times m}}
\]

\[
= \frac{0.5 \times $0.0237 + 0.5 \times $0.5024}{1.05}
\]
American put on coupon bond

- 1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays max[$0, K - P_i$]

\[
\begin{align*}
V_{0,0} &= ? \\
V_{1,0} &= ? \\
V_{1,1} &= ? \\
V_{2,0} &= \max[0, 99.00 - 100.00] = 0.00 \\
V_{2,1} &= \max[0, 99.00 - 99.05] = 0.00 \\
V_{2,2} &= \max[0, 99.00 - 98.11] = 0.89
\end{align*}
\]
American put on coupon bond

- 1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays max[ 0, $K - P_i$ ]

\[ V_{2,2} = \max[0,99.00 - 98.11] = 0.89 \]

\[ V_{2,1} = \max[0,99.00 - 99.05] = 0.00 \]

\[ V_{2,0} = \max[0,99.00 - 100.00] = 0.00 \]

\[ V_{1,1} = \frac{q_{1,1} \times V_{2,2} + (1 - q_{1,1}) \times V_{2,1}}{(1 + r_{1,1}/m)^{1 \times \Delta t \times m}} \frac{K - P_{1,1}}{\text{exercise}} \]

\[ = \max \left[ 0.5 \times 0.89 + 0.5 \times 0.00 \right] \frac{1.055}{\text{max}[0,99.00 - 97.23]} \]

\[ = \max \left[ 0.5 \times 0.89 + 0.5 \times 0.00 \right] \frac{1.055}{99.00 - 97.23} \]
American put on coupon bond

- 1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays max$[0, K - P_i]$.

$V_{0,0} = ?$

$V_{1,0} = \$0.00$

$V_{1,1} = \$1.7674$

$V_{2,0} = \max[0, \$99.00 - \$100.00] = \$0.00$

$V_{2,1} = \max[0, \$99.00 - \$99.05] = \$0.00$

$V_{2,2} = \max[0, \$99.00 - \$98.11] = \$0.89$

$V_{1,0} = \max \left[ q_{1,0} \times V_{2,1} + (1 - q_{1,0}) \times V_{2,0}, K - P_{1,0} \right]$

$= \max \left[ 0.5 \times \$0.00 + 0.5 \times \$0.00, \$99.00 - \$99.07 \right]$

$= \$0.00$
American put on coupon bond

- 1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays max\[ 0, K - P_i \]

\[
V_{0,0} = \max \left[ \frac{q_{0,0} \times V_{1,1} + (1 - q_{0,0}) \times V_{1,0}}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}}, K - P_{0,0} \right]
\]

\[
V_{0,0} = \max \left[ 0.5 \times 1.7674 + 0.5 \times 0.00, 99.00 - 97.28 \right]
\]

\[
V_{1,0} = 0.00
\]

\[
V_{1,1} = 1.7674
\]

\[
V_{2,0} = \max [0, 99.00 - 100.00] = 0.00
\]

\[
V_{2,1} = \max [0, 99.00 - 99.05] = 0.00
\]

\[
V_{2,2} = \max [0, 99.00 - 98.11] = 0.89
\]

\[
V_{0,0} = 1.7150
\]
Callable Bond

• Suppose we want to price a 10% 5-yr coupon bond callable (by the issuer) at the end of year 3 at par
  – Step 1: Determine the price of the non-callable bond, $P_{NCB}$
  – Step 2: Determine the price of the call option on the non-callable bond with expiration after 3 years and strike price at par, $O_{NCB}$
  – Step 3: The price of the callable bond is
    \[ P_{CB} = P_{NCB} - O_{NCB} \]

• Intuition
  – The bondholder grants the issuer an option to buy back the bond
  – The value of this option must be subtracted from the price the bondholder pays the issuer for the non-callable bond
Putable Bond

• Suppose we want to price a 10% 5-yr coupon bond putable (by the bondholder to the issuer) at the end of year 3 at par
  – Step 1: Determine the price of the non-putable bond, $P_{NPB}$
  – Step 2: Determine the price of the put option on the non-putable bond with expiration after 3 years and strike price at par, $O_{NPB}$
  – Step 3: The price of the putable bond is
    \[ P_{PB} = P_{NPB} + O_{NPB} \]

• Intuition
  – The bond issuer grants the holder an option to sell back the bond
  – The value of this option must be added to the price the bondholder pays the issuer for the non-putable bond
Spot rate process

- Binomial trees are based on spot rate values $r_{i,j}(1)$ and risk-neutral probabilities $q_{i,j}$

- In single-factor models, these values are determined by a risk-neutral spot rate process of the form

$$ r_{t+\Delta t}(1) - r_t(1) = \mu [r_t(1), t] \times \Delta t + \sigma [r_t(1), t] \times \sqrt{\Delta t} \times \epsilon_t $$

with

Mean[$\epsilon_t$] = 0 \hspace{1cm} Var[$\epsilon_t$] = 1

such that

Mean[$r_{t+\Delta t}(1) - r_t(1)$] = $\mu [r_t(1), t] \times \Delta t$

Var[$r_{t+\Delta t}(1) - r_t(1)$] = $\sigma [r_t(1), t]^2 \times \Delta t$
Spot rate process (cont)

• In an $N$-factor models, these values are determined by a risk-neutral spot rate process of the form

$$r_t(1) = z_{1,t} + z_{2,t} + \cdots + z_{N,t}$$

with

$$z_{i,t+\Delta t} - z_{i,t} = \mu_i \left[ z_{1,t}, z_{2,t}, \cdots, z_{N,t}, t \right] \times \Delta t +$$

$$\sigma_i \left[ z_{1,t}, z_{2,t}, \cdots, z_{N,t}, t \right] \times \sqrt{\Delta t} \times \epsilon_{1,t}$$

and

$$\text{Mean}[\epsilon_{i,t}] = 0 \quad \text{Var}[\epsilon_{i,t}] = 1$$
Drift function

- Case 1: Constant drift

\[ r_{t+\Delta t} - r_t = \lambda x \Delta t + \sigma x \sqrt{t} x \epsilon_t \]

with

\[ \epsilon_t \sim N[0, 1] \]

- Implied distribution of 1-period spot rates

\[ r_{t+\Delta t} \sim N\left[ r_t + \lambda \times \Delta t, \sigma^2 \times \Delta t \right] \]
Drift function (cont)

- Binomial tree representation

\[
q = 1/2 \quad r_{0,0} + \lambda \times \Delta t + \sigma \times v \Delta t
\]

\[
q = 1/2 \quad r_{0,0} + 2 \times \lambda \times \Delta t + 2 \times \sigma \times v \Delta t
\]

- Properties
  - No mean reversion
  - No heteroskedasticity
  - Spot rates can become negative, but not if we model \(\ln(r(1))\)
    \(\Rightarrow\) “Rendleman-Bartter model”
  - 2 parameters
  - \(\Rightarrow\) fit only 2 spot rates
### Drift function (cont)

- **Example**
  - \( r_{0,0} = 5\% \)
  - \( \lambda = 1\% \)
  - \( \sigma = 2.5\% \)
  - \( \Delta t = 1/m \) with \( m = 2 \)

\[
\begin{array}{l}
q = 1/2 \\
r_{0,0} = 5\% \\
r_{1,0} = 3.73\% \\
r_{1,1} = 7.27\% \\
r_{2,0} = 2.47\% \\
r_{2,1} = 6.00\% \\
r_{2,2} = 9.54\% \\
\end{array}
\]
Drift function (cont)

• **Case 2: Time-dependent drift**

\[
\begin{align*}
\lambda_{t+\Delta t} - \lambda_t & = \lambda(t) \times \Delta t + \sigma \times \sqrt{t} \times \epsilon_t \\
& \text{drift fct}
\end{align*}
\]

with

\[
\epsilon_t \sim N[0, 1]
\]

• Implied distribution of 1-period spot rates

\[
\begin{align*}
\lambda_{t+\Delta t} & \sim N \left[ \lambda_t + \lambda(t) \times \Delta t, \sigma^2 \times \Delta t \right] \\
& \text{mean/var}
\end{align*}
\]

• Ho and Lee (1986, *J. of Finance*) ⇒ “Ho-Lee model”
Drift function (cont)

- Binomial tree representation

\[
\begin{align*}
q = 1/2 & \
\begin{cases}
q = 1/2 & r_{0,0} + \lambda(1) \times \Delta t + \sigma \times \nu \Delta t \\
q = 1/2 & r_{0,0} + [\lambda(1) + \lambda(2)] \times \Delta t + 2 \times \sigma \times \nu \Delta t
\end{cases}
\end{align*}
\]

- Properties
  - No heteroskedasticity
  - Spot rates can become negative, but not if we model \(\ln(r(1))\) ⇒ “Salomon Brothers model”
  - Arbitrarily many parameters
    ⇒ fit term structure of spot rates but not necessarily spot rate volatilities (i.e., derivative prices)
Drift function (cont)

• Case 3: Mean reversion

\[ r_{t+\Delta t} - r_t = \kappa \times [\theta - r_t(1)] \times \Delta t + \sigma \times \sqrt{\Delta t} \times \epsilon_t \]

with

\[ \epsilon_t \sim N[0, 1] \]

• Implied distribution of 1-period spot rates

\[ r_{t+\Delta t} \sim N \left[ r_t + \kappa \times [\theta - r_t(1)] \times \Delta t, \sigma^2 \times \Delta t \right] \]

• Vasicek (1977, J. of Financial Economics) \( \Rightarrow \) “Vasicek model”
Drift function (cont)

• Binomial tree representation

\[
\begin{align*}
q = \frac{1}{2} & \quad r_{0,0} + \kappa \times (\theta - r_{0,0}) \times \Delta t + \sigma \times v \times \Delta t \\
q = \frac{1}{2} & \quad r_{1,1} + \kappa \times (\theta - r_{1,1}) \times \Delta t + \sigma \times v \times \Delta t \\
q = \frac{1}{2} & \quad r_{1,0} + \kappa \times (\theta - r_{1,0}) \times \Delta t - \sigma \times v \times \Delta t
\end{align*}
\]

• Properties
  – Non-recombining, but can be fixed
  – No heteroskedasticity
  – Spot rates can become negative, but not if we model \( \ln[r(1)] \)
  – 3 parameters
    \( \Rightarrow \) fit only 3 spot rates
Drift function (cont)

• Example
  
  − $r_{0,0} = 5\%$
  − $\theta = 10\%$
  − $\kappa = 0.25$
  − $\sigma = 2.5\%$
  − $\Delta t = 1/m$ with $m = 2$

  With $\kappa = 0$
  
  $r_{2,0} = 8.54\%$

  $r_{2,1} = 5.00\%$
  
  $r_{2,1} = 5.00\%$

  $r_{2,2} = 9.49\%$
  
  $r_{2,2} = 8.54\%$

  $r_{1,1} = 7.39\%$

  $r_{2,1} = 5.95\%$
  
  $r_{2,1} = 5.00\%$

  $r_{1,1} = 7.39\%$

  $r_{2,1} = 6.39\%$
  
  $r_{2,1} = 5.00\%$

  $r_{1,0} = 3.86\%$

  $r_{2,0} = 2.86\%$
  
  $r_{2,0} = 1.46\%$
Drift function (cont)

• Example

- $r_{0,0} = 15\%$
- $\theta = 10\%$
- $\kappa = 0.25$
- $\sigma = 2.5\%$
- $\Delta t = 1/m$ with $m = 2$

\[
\begin{align*}
q = 1/2 & \quad r_{2,2} = 17.14\% \\
q = 1/2 & \quad r_{1,1} = 16.77\% \\
r_{0,0} = 15\% & \quad r_{2,2} = 18.54\% \\
r_{0,0} = 15\% & \quad r_{2,2} = 18.54\% \\
r_{1,0} = 12.61\% & \quad r_{2,1} = 14.05\% \\
r_{1,0} = 12.61\% & \quad r_{2,1} = 15.00\% \\
r_{1,0} = 12.61\% & \quad r_{2,1} = 15.00\% \\
r_{1,0} = 12.61\% & \quad r_{2,0} = 10.51\% \\
r_{1,0} = 12.61\% & \quad r_{2,0} = 11.46\% \\
\end{align*}
\]

With $\kappa = 0$
Volatility function

- Case 1: Square-root volatility

\[ r_{t+\Delta t} - r_t = \lambda \times \Delta t + \sigma \times \sqrt{r_t} \times \sqrt{t} \times \epsilon_t \]

with

\[ \epsilon_t \sim N[0, 1] \]

- Implied distribution of 1-period spot rates

\[ r_{t+\Delta t} \sim N\left[ r_t + \lambda \times \Delta t, \sigma^2 \times r_t \times \Delta t \right] \]

- Cox, Ingersoll, and Ross (1985, *Econometrics*) ⇒ “CIR model”
Volatility function (cont)

- Binomial tree representation

\[
q = \frac{1}{2} \quad r_{1,1} + \lambda \Delta t + \sigma v r_{0,0} \Delta t
\]
\[
q = \frac{1}{2} \quad r_{1,0} + \lambda \Delta t - \sigma v r_{0,0} \Delta t
\]
\[
r_{0,0} + \lambda \Delta t + \sigma v r_{0,0} \Delta t
\]
\[
r_{0,0} + \lambda \Delta t - \sigma v r_{0,0} \Delta t
\]

- Properties
  - Non-recombining, but can be fixed
  - No mean-reversion, but can be fixed by using different drift function
  - Spot rates can become negative, but not as \( \Delta t \to 0 \)
  - 1 volatility parameter (and arbitrarily many drift parameters)
    \( \Rightarrow \) fit term structures of spot rates but only 1 spot rate volatility
Volatility function (cont)

• Example

- \( r_{0,0} = 5\% \)
- \( \lambda = 1\% \)
- \( \sigma = 11.18\% \) \( \Rightarrow \sigma \times v r_{0,0} = 2.5\% \)
- \( \Delta t = 1/m \) with \( m = 2 \)

\[
\begin{align*}
q &= \frac{1}{2} \\
q &= \frac{1}{2} \\
r_{0,0} &= 5\% \\
r_{1,0} &= 3.73\% \\
r_{1,1} &= 7.27\% \\
r_{2,0} &= 2.70\% \\
r_{2,1} &= 5.56\% \\
r_{2,2} &= 9.90\%
\end{align*}
\]

With constant volatility

- \( r_{2,0} = 2.47\% \)
- \( r_{2,1} = 0.60\% \)
- \( r_{2,2} = 9.54\% \)
Volatility function (cont)

• Case 2: Time-Dependent volatility

\[ r_{t+\Delta t} - r_t = \lambda \times \Delta t + \sigma(t) \times \sqrt{t} \times \epsilon_t \]

with

\[ \epsilon_t \sim N[0, 1] \]

• Implied distribution of 1-period spot rates

\[ r_{t+\Delta t} \sim N \left( \left[ r_t + \lambda \times \Delta t, \sigma(t)^2 \times \Delta t \right] \right) \]

• Hull and White (1993, *J. of Financial and Quantitative Analysis*)

\[ \Rightarrow \text{“Hull-White model”} \]
Volatility function (cont)

• Binomial tree representation

\[
\begin{align*}
q &= \frac{1}{2} & r_{1,1} &= r_{0,0} + \lambda \times \Delta t + \sigma(1) \times \nu \Delta t \\
q &= \frac{1}{2} & r_{1,0} &= r_{0,0} + \lambda \times \Delta t - \sigma(1) \times \nu \Delta t \\
q &= \frac{1}{2} & r_{0,1} &= r_{1,0} + \lambda \times \Delta t + \sigma(2) \times \nu \Delta t \\
q &= \frac{1}{2} & r_{0,0} &= r_{1,1} + \lambda \times \Delta t - \sigma(2) \times \nu \Delta t
\end{align*}
\]

• Properties
  – Non-recombining, but can be fixed
  – No mean-reversion, but can be fixed by using different drift function
  – Spot rates can become negative, but not if we model \( \ln[r(1)] \)
    \( \Rightarrow \) “Black-Karasinski model” and “Black-Derman-Toy model”
  – Arbitrarily many volatility and drift parameters
    \( \Rightarrow \) fit term structures of spot rates and volatilities
Calibration

- To calibrate parameters of a factor model to bonds prices
  - Step 1: Pick arbitrary parameter values
  - Step 2: Calculate implied 1-period spot rate tree
  - Step 3: Calculate model prices for liquid securities
  - Step 4: Calculate model pricing errors given market prices
  - Step 5: Use solver to find parameter values which minimize the sum of squared pricing errors
Constant drift example

• Step 1: Pick arbitrary parameter values

Parameters

?  0.00%
s  0.10%

Observed 1-period spot rate

r(1)  6.21%
Constant drift example (cont)

• Step 2: Calculate implied 1-period spot rate tree

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**Constant drift example (cont)**

- **Step 3: Calculate model prices for liquid securities**
  - E.g., for a 2.5-yr STRIPS

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### Constant drift example (cont)

- Step 3: Calculate model prices for liquid securities (cont)

<table>
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<tr>
<th>Parameters</th>
<th>Model implied</th>
<th>Periods</th>
<th>spot rate</th>
</tr>
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<tr>
<td>?</td>
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<td>s</td>
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<td>6.21%</td>
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<tr>
<td>Observed 1-period spot rate</td>
<td>2.5</td>
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## Constant drift example (cont)

- **Step 4: Use solver**

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<th>Model implied spot rate</th>
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<th>Pricing error</th>
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<td>6.21%</td>
<td>6.97%</td>
<td>0.76%</td>
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</tbody>
</table>

**Observed 1-period spot rate**

- **Minimize sum of squared errors by choice of parameters**

**Sum of squared errors** 0.0002521
### Constant drift example (cont)

- **Solution**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model implied spot rate</th>
<th>Observed spot rate</th>
<th>Pricing error</th>
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</thead>
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<td>( \phi )</td>
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<tr>
<td>( s )</td>
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<tr>
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<tr>
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<td>6.71%</td>
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<td>( r(1) )</td>
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<td>6.81%</td>
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</table>

Sum of squared errors \( 7.882E-07 \)