Final

MS&E 310

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Question 1 (20pts)

Consider the combinatorial call auction market discussed in the class. This time, the market maker forms the decision problem as:

$$\begin{array}{lll} \max & \sum_{j=1}^{n} x_j & \\ \text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot y & \leq \mathbf{0}, \ (\mathbf{p}) \\ & -\pi^T \mathbf{x} + \alpha \cdot y & \leq 0, \ (\lambda) \\ & \mathbf{x} & \leq \mathbf{q}, \ (\mu) \\ & \mathbf{x} & \geq \mathbf{0}, \end{array}$$

where $(\pi_j, \mathbf{a}_j, q_j)$ are as defined as in our auction problem through out this course, $\mathbf{a}_j$ is the column vector of A, $\mathbf{e}$ is the vector of all ones, and parameter $\alpha \geq 0$. Again, the bidder wins one dollar if the winning state is in his or her selection.

(a) (5 pts) Interpret the model, and write down the dual of the model. In particular, what is the role of α?

(b) (10 pts) Use the dual variable indicated in the formulation to derive the dual linear program and the optimal conditions. Show that $\mathbf{p} = \mathbf{0}$ and $\lambda = 0$ is a dual feasible solution

(c) (5 pts) Show by an example that the model may have an arbitrage, that is, an order with $\mathbf{a}_j = \mathbf{e}$ and price limit $\pi_j < 1$ can get accepted.

Answer

Solution: (a) This model tries to accept the maximum amount of bids while the market maker makes sure that the worst-case ratio of revenue over cost is at least $\alpha > 0$.

The dual of the model is

$$
\begin{array}{ll}
\text{minimize}_{p,\lambda,\mu} & q^T\mu \\
\text{subject to} & A^Tp - \lambda\pi + \mu \geq e \\
& e^Tp = \alpha \cdot \lambda \\
& p, \lambda, \mu \geq 0
\end{array}
\tag{1}
$$

(b) The dual formulation is given in 1). The optimality conditions for the primal and dual are:

- Primal Feasible

$$
\begin{aligned}
Ax - e \cdot y &\leq 0 \\
-\pi^T x + \alpha \cdot y &\leq 0 \\
x &\leq q \\
x &\geq 0
\end{aligned}
$$

- Dual Feasible

$$
\begin{aligned}
A^Tp - \lambda\pi + \mu &\geq e \\
e^Tp &= \alpha \cdot \lambda \\
p, \lambda, \mu &\geq 0
\end{aligned}
$$

- Zero Duality Gap

$$e^Tx = q^T\mu$$

  or the complementary slackness

$$
\begin{aligned}
x^T(A^Tp - \lambda\pi + \mu - e) &= 0 \\
p^T(Ax - e \cdot y) &= 0 \\
\lambda(-\pi^T x + \alpha \cdot y) &= 0 \\
\mu^T(q - x) &= 0
\end{aligned}
$$

Now we show the dual feasibility of $p = 0$ and $\lambda = 0$. In this case $e^Tp = 0 = \alpha \cdot \lambda$, thus the second condition holds. The first condition $A^Tp - \lambda\pi + \mu \geq e$ reduces to $\mu \geq e$. We can set, say, $\mu = e$, then all dual feasible constraints are satisfied.

(c) Let $\alpha = 1$. Consider an example with two states and three bids, where $a_1 = (1;\ 0)$, $a_2 = (0;\ 1)$ and $a_3 = (1;\ 1)$, $\pi_1 = \pi_2 = \pi_3 = 0.6$, and $q_1 = q_2 = 2$ and $q_3 = 1$. It is easy to see that $y = 3$, $x_1 = x_2 = 2$ and $x_3 = 1$ is the optimal solution. This is an arbitrage, that is, Bidder 3 is for sure to win one dollar with cost sixty cents.

The optimal dual solution is $p_1 = p_2 = \lambda = 0$, and $\mu = e$.

Question 2 (20 pts)

Consider a (bounded) polytope in n-dimension defined by the following set of linear constraints

$$\{\mathbf{x}\ |A\mathbf{x} \le \mathbf{b}\}.$$

Suppose that we wish to embed this polytope in the "smallest" possible square whose sides are parallel to the coordinate axes. One way to do is for each j to solve max and min values of the following linear programs

$$\bar{x}_j := \max\ x_j \quad \text{s.t.} \quad A\mathbf{x} \le \mathbf{b};$$

$$\underline{x}_j := \min\ x_j \quad \text{s.t.} \quad A\mathbf{x} \le \mathbf{b}.$$

Then you let the jth coordinate of the center of the square be $(\bar{x}_j + \underline{x}_j)/2$ where each side has $\max_j\ (\bar{x}_j - \underline{x}_j)$ in length. In doing so, you need to solve $2n$ linear programs.

(a) (10pts) How to formulate the problem as a *single* linear program. (Hint. Consider to use the dual.)

(b) (10pts) Let

$$A = \begin{pmatrix} 1 & 3 & 2 \\ -2 & -1 & -4 \\ -1 & 2 & 1 \\ 2 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 3 \\ 1 \end{pmatrix}.$$

Use any method to find such "smallest" square.

Answer

Solution

(a) For the first type of problem $\min x_i$ s.t. $Ax \leq b$, it has the same optimal objective value as the dual, which is:

$$\begin{array}{ll} \max & b^T y^{(i)} \\ \text{s.t.} & A^T y^{(i)} = e_i \\ & y^{(i)} \leq 0 \end{array}$$

Here, e_i is the i-th Euclidean basis vector in R^n and $y^{(i)} \in R^m$. Introducing an extra variable a_i (which should be equal to $\underline{x}_i$ at optimality), we have:

$$\begin{array}{ll} \max_{a_i, y^{(i)}} & a_i \\ \text{s.t.} & a_i \leq b^T y^{(i)} \\ & A^T y^{(i)} = e_i \\ & y^{(i)} \leq 0 \end{array}$$

For the second type of problem $\max x_i$ s.t. $Ax \leq b$, it has the same optimal objective value as the dual, which is:

$$\begin{array}{ll} \min & b^T z^{(i)} \\ \text{s.t.} & A^T z^{(i)} = e_i \\ & z^{(i)} \geq 0 \end{array}$$

Here, $z^{(i)} \in R^m$. Introducing an extra variable d_i (which should be equal to $\bar{x}_i$ at optimality), we have:

$$\begin{array}{ll} \min_{d_i, z^{(i)}} & d_i \\ \text{s.t.} & b^T z^{(i)} \leq d_i \\ & A^T z^{(i)} = e_i \\ & z^{(i)} \geq 0 \end{array}$$

We can turn it into a maximization problem by negating the objective function. Then, we can combine the $2n$ problems into one LP:

$$\begin{array}{cl} \max_{a_i, d_i, x^{(i)}, y^{(i)} \text{ for } i=1,\ldots,n} & \sum_{i=1}^n a_i - d_i \\ \text{s.t.} & a_i \le b^T y^{(i)} \text{ for } i = 1, \ldots, n \\ & A^T y^{(i)} = e_i \text{ for } i = 1, \ldots, n \\ & b^T z^{(i)} \le d_i \text{ for } i = 1, \ldots, n \\ & A^T z^{(i)} = e_i \text{ for } i = 1, \ldots, n \\ & y^{(i)} \le 0 \text{ for } i = 1, \ldots, n \\ & z^{(i)} \ge 0 \text{ for } i = 1, \ldots, n \end{array}$$

(b) Solving the final LP above, we obtain $a = \begin{bmatrix} -\frac{22}{3} \\ -6 \\ \frac{1}{3} \end{bmatrix}$ and $d = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{17}{3} \end{bmatrix}$.
Hence, the smallest possible rectangle for this example is $[-\frac{22}{3}, \frac{2}{3}] \times [-6, 0] \times [\frac{1}{3}, \frac{17}{3}]$. So the square edge length is 8.

Question 3 (20 pts)

In this question, we will reconsider the relaxation problem in midterm. Given a function

$$\phi(\mathbf{x}) = \boldsymbol{\alpha}^T \mathbf{x} + \frac{1}{2}\mathbf{x}^\mathbf{T}\mathbf{W}\mathbf{x} = \sum_i \alpha_i x_i + \sum_{i<j} w_{ij} x_i x_j,$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)$ is a binary vector ($x_i \in \{-1, 1\}$), $\boldsymbol{W}_{ij} = \boldsymbol{W}_{ji}$ and $\boldsymbol{W}_{ii} = 0$. Our goal is to

$$\text{(D-OPT)} \ \max_{\mathbf{x}} \ \ \phi(\mathbf{x}),$$

$$\text{subject to } x_i \in \{-1, 1\}.$$

(a) (10 pts) Please provide a semidefinite programming (SDP) relaxation for the above problem. (Hint. You may want to introduce a matrix $\boldsymbol{Y} = (y_{i,j})_{n,n}$ to linearize the quadratic part in the objective function.)

Answer

$$\max_{\mathbf{x}, \mathbf{Y}} \ \ \boldsymbol{\alpha}^T \mathbf{x} + \sum_{i<j} w_{ij} y_{ij},$$

$$\text{subject to} \quad y_{i,j} = y_{j,i}, \; y_{i,i} = 1, \quad \boldsymbol{Y} \succeq \boldsymbol{x}\boldsymbol{x}^T.$$

(b) (10 pts) Write out the dual problem of your SDP formulation.

Answer

Rewrite the primal problem as (by following the tricks in max-cut problem in lecture 15):

$$\max_{\mathbf{X}} \quad \sum_{i=2}^{n+1} \alpha_i Z_{1,i} + \sum_{2 \leq i < j \leq n+1} W_{i,j} Z_{i+1,j+1},$$

$$\text{subject to } z_{i,i} = 1, z_{i,j} = z_{j,i} \quad \text{for } i \neq j$$

$$\boldsymbol{Z} \succeq 0.$$

Then by replacing the coefficients in the objective with C and constraints as $\mathcal{A}$, its dual follows from the Page 6 of lecture 14.

Question 4 (20 pts + 10 pts Bonus)

Consider the optimization problem

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \|\boldsymbol{A}\mathbf{x} - \mathbf{b}\|_1,$$

where $\mathbf{x} \in R^n$ is the decision variable vector, $\boldsymbol{A} \in R^{m \times n}$, $\boldsymbol{b} \in R^m$ and $\| \cdot \|_1$ is the L^1-norm. For $\mathbf{u} = (u_1, ..., u_k) \in R^k$, $\|\mathbf{u}\|_1 = \sum_{i=1}^k |u_i|$. We assume that $m < n$. This optimization problem has a background compressed sensing, a signal processing technique for efficient signal reconstruction. Imagine we have a signal $\mathbf{x}_{\text{true}} \in R^n$, you can see each column of matrix $\boldsymbol{A}$ as a measurement. Normally, we need n measurements to reconstruct the true signal. However, when there are some sparsity conditions (number of non-zero elements) with $\mathbf{x}_{\text{true}}$, we could actually recover it by solving the above optimization problem (in which there are only $m < n$ measurements). In other words, the optimal solution of the above optimization problem $\mathbf{x}^* = \mathbf{x}_{\text{true}}$.

(a) (15 pts) Reformulate the optimization problem into an LP and provide the path following algorithm for your formulation.

Answer

The LP formulation is

$$\min_{u,v,s,t} \quad \sum_{i=1}^{n}(u_i+v_i)+\sum_{i=1}^{m}(s_i+t_i),$$

$$Au-Av-b=s-t,$$

$$u,v\in R_+^n,\quad s,t\in R_+^m.$$

And the path following algorithm would be

$$\min_{u,v,s,t} \quad \sum_{i=1}^{n}(u_i+v_i)+\sum_{i=1}^{m}(s_i+t_i)+\mu\sum_{i=1}^{n}(\log u_i+\log v_i)+\mu\sum_{i=1}^{m}(\log s_i+\log t_i)$$

$$Au-Av-b=s-t.$$

And we alternatively decrease μ and optimize over u,v,s,t.

(b) (5 pts) What is the augmented Lagrangian function for you formulation?

Answer

The augmented Lagrangian is

$$\min_{u,v,s,t} \quad \sum_{i=1}^{n}(u_i+v_i)+\sum_{i=1}^{m}(s_i+t_i)-y^T(Au-Av-b-s+t)+\frac{\beta}{2}\|Au-Av-b-s+t\|_2^2,$$

$$u,v\in R_+^n,\quad s,t\in R_+^m,\quad y\in R^m.$$

(c) (Bonus 10 pts) ADMM algorithm has been a benchmark algorithm for solving compressed sensing problem. Provide ADMM algorithm for your formulation. Furthermore, we require that the ADMM should have analytical (i.e. closed-form) updates on each step. Is the ADMM a 2-block or multi-block one? Discuss about the convergence.

Answer

Introduce variables to separate out the non-negative constraints. The problem will become:

$$\min_{u,v,s,t,u',v',s',t'} \quad \sum_{i=1}^{n}(u_i + v_i) + \sum_{i=1}^{m}(s_i + t_i),$$

$$Au - Av - b = s - t,$$

$$u = u', v = v', s = s', t = t'$$

$$u', v' \in R^n_+, \quad s', t' \in R^m_+.$$

The augmented Lagrangian is

$$\begin{aligned}
\min_{u,v,s,t,u',v',s',t'} \quad & \sum_{i=1}^{n}(u_i + v_i) + \sum_{i=1}^{m}(s_i + t_i) - y^T(Au - Av - b - s + t) \\
& + \frac{\beta}{2}\|Au - Av - b - s + t\|_2^2 + \frac{\beta}{2}\|u - u'\|_2^2 + \frac{\beta}{2}\|v - v'\|_2^2 \\
& + \frac{\beta}{2}\|s - s'\|_2^2 + \frac{\beta}{2}\|t - t'\|_2^2 - z_u^T(u - u') - z_v^T(v - v') \\
& - z_s^T(s - s') - z_t^T(t - t'), \\
\text{subjec to} \quad & u', v' \in R^n_+, \quad s', t' \in R^m_+
\end{aligned}$$

It is essentially a 2-block ADMM. One group is (u, v, s, t) and the other one is (u', v', s', t'). The rest are all dual variables. In the objective function, the first group is quadratic free variables, while the second group is quadratic and decomposable (but subject to positive constraints). So both groups have analytical updates.