MS&E 310 | **Homework #1**
Linear Programming | **Optional - not graded**

HOMEWORK ASSIGNMENT 1

Reading: Lecture Notes 1, 2, 3 and 4, and Chapters 1, 2 and 3.1-3.2 of the text book.

1. Suppose there are 5 securities available in the World Cup Assets market for open trading at fixed prices and pay-offs; see the table below. Here, for example, Security 1's pay-off is $1 if either Argentina, Brazil, or Italy wins. The Share Limit represent the maximum number of shares one can purchase, and Price is the current purchasing price per share of each security.

Security	Price π	Share Limit q	Argentina	Brazil	Italy	Germany	France
1	$0.75	10	$1	$1	$1		
2	$0.35	5				$1	
3	$0.40	10	$1		$1		$1
4	$0.95	10	$1	$1	$1	$1	
5	$0.75	5		$1		$1	

1. Assume that short is not allowed, that is, one can only buy shares not sell. Formulate the problem to decide how many shares of each security to purchase so as to maximize the worst-case (minimum) pay-off when the game is finally realized.

2. Now assume there is no share limit (can be ∞) and short is allowed, that is, the decision variable can be both positive (buy) and negative (sell). Reformulate the portfolio problem to decide how many shares of each security to purchase so as to maximize the worst-case (minimum) pay-off when the game is finally realized.

2. Exercise 2.8-9.

3. Exercise 2.8-10.

4. Exercise 2.8-13.

5. Exercise 2.8-16.

6. Consider the two-variable linear program with 6 inequality constraints:

$$\begin{array}{ll} \text{maximize} & 3x_1 + 5x_2 \\ \text{subject to} & x_1 \geq 0 \\ & x_2 \geq 0 \\ & -x_1 + x_2 \leq 2.5 \\ & x_1 + 2x_2 \leq 9 \\ & x_1 \leq 4 \\ & x_2 \leq 3 \end{array}$$

(a) Plot the lines $x_j = 0 \quad j = 1, \dots 6$ (all on the same two-dimensional graph of x_1 and x_2) where, for $j = 3, 4, 5, 6$, x_j denotes the slack variable in the j^{th} constraint.

(b) Identify the extreme points of the feasible region as intersections of suitable lines $x_j = 0$.

(c) For each pair of *adjacent* extreme points of the feasible region, describe how each direction of the edge between them can be generated by increasing the value of a single variable chosen from $\{x_1, x_2, x_3, x_4, x_5, x_6\}$.

7. Consider the standard form polyhedron $P = \{\mathbf{x} : \; A\mathbf{x} = \mathbf{b}, \; \mathbf{x} \geq \mathbf{0}\}$. Suppose that the matrix A has dimensions $m \times n$ and that its rows are linearly independent. For each one of the following statements, state whether it is true or false. If true, provide a proof, else, provide a counterexample.

1. If $n = m + 1$, then P has at most two basic feasible solutions.
2. The set of all optimal solutions is bounded.
3. At every optimal solution, no more than m variables can be positive.
4. If there is more than one optimal solution, then there are uncountably many optimal solutions.
5. If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.
6. Consider the problem of minimizing $\max\{\mathbf{c}^T\mathbf{x} + c_0, \mathbf{d}^T\mathbf{x} + d_0\}$ over the set P. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of P.

8. Convexity:

1. Assuming the following linear conic program is strictly feasible, we consider the minimal–objective function $\mathbf{c} \mapsto z(\mathbf{c})$ for fixed A and $\mathbf{b}$:

$$
\begin{array}{rll}
z(\mathbf{c}) := & \text{minimize} & \mathbf{c}^T\mathbf{x} \\
& \text{subject to} & A\mathbf{x} = \mathbf{b} \\
& & \mathbf{x} \in K
\end{array}
$$

   for a closed convex cone $K \subset R^n$. Show that $z(\cdot)$ is a concave function in the vector $\mathbf{c}$ where $z(\mathbf{c})$ is finite and attained at an optimal solution.

2. Let $\mathbf{c}$ be any given n-dimensional vector and $\lambda_k(\mathbf{c})$ be the $k-th$ smallest element of $\mathbf{c}$, where $k < n$. Then, prove the following statement

$$
\begin{array}{rll}
\lambda_1(\mathbf{c}) + \cdots \lambda_k(\mathbf{c}) = & \text{minimize} & \mathbf{c}^T\mathbf{x} \\
& \text{subject to} & \mathbf{e}^T\mathbf{x} = k, \\
& & \mathbf{0} \le \mathbf{x} \le \mathbf{e}.
\end{array}
$$

   Thus, the *sum of k smallest elements function* of $\mathbf{c}$ is a concave function in $\mathbf{c}$.

3. Let $f(\mathbf{x})$ be a convex function. Then, prove that $\phi(t, \mathbf{x}) = t \cdot f(\mathbf{x}/t)$ is a convex function in $\mathbf{x}$ and $t > 0$.

9. Suppose $\mathbf{v} = (v_1, ..., v_m)$ is an arbitrary vector with positive components. Show that the following linear program also gives the solution to the MDP problem on Slide 30 Lecture 1.

$$
\begin{array}{ll}
\text{maximize}_{\mathbf{y}} & \displaystyle\sum_{i=1}^{m} v_i y_i \\
\text{subject to} & y_i - \gamma \mathbf{p}_j \mathbf{y} \le c_j \quad \text{for } j \in \mathcal{A}_i \text{ and } i = 1, ..., m
\end{array}
$$

In other words, the above LP shares the same optimal solution as the LP on on Slide 31 Lecture 1.