MS&E 310 **Homework #2 Solution**
Linear Programming

HOMEWORK ASSIGNMENT 2 SOLUTION

1. Farkas' lemma can be used to derive many other (named) theorems of the alternative. This problem concerns a few of these pairs of systems. Using Farkas's lemma, prove each of the following results.

(a) Gordan's Theorem. Exactly one of the following systems has a solution:

$$\begin{array}{ll} \text{(i)} & Ax > 0 \\ \text{(ii)} & y^T A = 0, \quad y \geq 0, \quad y \neq 0. \end{array}$$

Let b be any fixed positive vector (e.g. the all-one vector). Then, (i) has a solution iff $Ax \geq b$ has a solution: in fact, if $Ax > 0$, then one can always scale x so that $Ax \geq b$. We can write $Ax \geq b$ as:

$$Ax' - Ax'' - z = b, \ (x'; x''; z) \geq 0$$

By Farkas' lemma, if it has no solution, then we must have an y such that:

$$y^T(A, \ -A, \ -I) \leq 0, \ y^T b = 1$$

Then, y satisfies (ii). Conversely, if (ii) has no solution, then the above system has no solution, and thus $Ax \geq b$ has a solution.

(b) Stiemke's Theorem. Exactly one of the following systems has a solution:

$$\begin{array}{ll} \text{(i)} & Ax \geq 0, \quad Ax \neq 0 \\ \text{(ii)} & y^T A = 0, \quad y > 0 \end{array}$$

Let b be any fixed positive vector (e.g. the all-one vector). Then, (i) is equivalent to $Ax \geq 0, \ b^T Ax = 1$ and it can be written as:

$$\begin{pmatrix} A & -A & -I \\ b^T A & -b^T A & 0 \end{pmatrix} (x'; x''; z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ (x'; x''; z) \geq 0 \qquad (1)$$

By Farkas' lemma, if it has no solution, then we must have a pair $(y';\tau)$ *such that:*

$$(y';\tau)^T \begin{pmatrix} A & -A & -I \\ b^T A & -b^T A & 0 \end{pmatrix} \leq 0, \ (y';\tau)^T(0;1) = 1 \tag{2}$$

Let $y = y' + \tau \cdot b$. *Then,* y *satisfies (ii). Conversely, if (ii) has no solution, then* $y^T A = 0, y \geq b$ *has no solution, which means* $y^T A = 0, y = y' + b, y' \geq 0$ *has no solution. Thus* (2) *has no solution; by Farkas' lemma,* (1) *has a solution, thus (i) has a solution.*

(c) Gale's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad Ax \leq b$$
$$\text{(ii)} \quad y^T A = 0, \quad y^T b < 0, \quad y \geq 0$$

Note that (i) can be written as:

$$Ax' - Ax'' + z = b, \ (x';x'';z) \geq 0$$

By Farkas' lemma, if it has no solution, then we must have an y *such that:*

$$y^T(A, \ -A, \ I) \leq 0, \ y^T b = 1$$

Then, $-y$ *satisfies (ii). The other direction is similar.*

2.

(a)

$$\begin{array}{ll} \text{minimize} & y^T b \\ \text{subject to} & y^T A \geq c^T \\ & y \text{ free.} \end{array}$$

(b)

$$\begin{array}{ll} \text{maxmize} & y^T b \\ \text{subject to} & y^T A \leq c^T \\ & y \geq 0. \end{array}$$

(c)

$$\begin{array}{ll} \text{maxmize} & y^T b + s^T \bar{b} \\ \text{subject to} & y^T A + s^T \bar{A} \leq c^T \\ & y \text{ free} \\ & s \geq 0. \end{array}$$

3. The LP model can be as

$$\begin{array}{ll} \text{maxmize} & \pi^T x - v^T z \\ \text{subject to} & Ax - z \le 0 \\ & x \le \bar{v} \\ & (x, z) \ge 0. \end{array}$$

where decision variables $x \in R^n$ and $z \in R^m$, and $\bar{v}$ is the quantity limit.

The dual of the problem is

$$\begin{array}{lrl} \min & \bar{v}^T y & \\ \text{s.t.} & A^T p + y & \ge \pi, \\ & p & \le v, \\ & (p, y) & \ge 0. \end{array}$$

The complementarity conditions imply that Most of these conditions are the same

$x_j > 0$	$a_j^T p + y_j = \pi_j$ so that $a_j^T p \le \pi_j$
$0 < x_j < \bar{v}_j$	$y_j = 0$ as well so that $a_j^T p = \pi_j$
$x_j = 0$	$y_j = 0$ so that $a_j^T p \ge \pi_j$
$z_i > 0$	$p_j = v_j$
$z_i = 0$	$p_j \le v_j$

as those in our earlier model. One interesting case is when $z_i = 0$, that is, nobody bid on state i, the price for that state can be any number between 0 and v_i. One simple case is let $p = v$ and we still have $e^T p = 1$.

4.

1. $$\begin{array}{lll} (LD) & \text{minimize} & b^T y + e^T s \\ & \text{subject to} & A^T y + s \ge c, \; y, s \ge 0. \end{array}$$

   $y_i : i = 1, 2, \cdots, m$: price for item i which has inventory b_i;

   $s_j : j = 1, 2, \cdots, n$: the difference between customer j's internal cost and external revenue.

2. Assume x, p, s is a strictly complementary solution.

   The strictly complementarity conditions imply that

$1 > x_j > 0$	$a_j^T y + s_j = c_j$ and $s_j = 0$ so that $a_j^T y = c_j$
$x_j = 0$	$a_j^T y + s_j > c_j$ and $s_j = 0$ so that $a_j^T y > c_j$
$x_j = 1$	$a_j^T y + s_j = c_j$ and $s_j > 0$ so that $a_j^T y < c_j$

3. Since the linear program pair has a strictly complementary primal solution x^* such that $x_j^* = 0$ or $x_j^* = 1$ for all j. The correctness of the mechanism follows directly from part (b).

5. Consider a system of m linear equations in n nonnegative variables, say

$$Ax = b, \quad x \geq 0.$$

Assume the right-hand side vector b is nonnegative. Now consider the (related) linear program

$$\begin{array}{ll} \text{minimize} & e^T y \\ \text{subject to} & Ax + Iy = b \\ & x \geq 0, \ y \geq 0 \end{array}$$

where e is the m-vector of all ones, and I is the $m \times m$ identity matrix. This linear program is called a *Phase One Problem.*

(a) Write the dual of the Phase One Problem.

$$\begin{array}{ll} \text{maximize} & b^T \pi \\ \text{subject to} & A^T \pi \leq 0 \\ & \pi \leq e \\ & \pi \text{ free} \end{array}$$

(b) Show that the Phase One Problem always has a basic feasible solution.

Obviously $[x; y] = [0; b]$ is a basic solution to the Phase One Problem; since b is nonnegative by the assumption, it is also a feasible solution.

(c) Using theorems proved in class, show that the Phase One Problem always has an optimal solution.

Since the Phase I problem is feasible, and its objective value is bounded from below by 0 or the dual of Phase I is feasible.

(d) Write the complementary slackness conditions for the Phase One Problem.

$$x_j(-A^T\pi)j = 0 \;\forall\; j = 1, ..., n$$
$$y_i(1-\pi_i) = 0 \;\forall\; i = 1, ..., m.$$

(e) Prove that $\{x : Ax = b, \quad x \geq 0\} \neq \emptyset$ if and only if the optimal value of the objective function in the corresponding Phase One Problem is zero.

If the optimal value of the Phase one problem is zero, then we must have also the optimal solution $(x \geq 0, y = 0)$ and that $Ax = b$, that is, $\{x : Ax = b, \quad x \geq 0\} \neq \emptyset$. Conversely, if $\{x : Ax = b, \quad x \geq 0\} \neq \emptyset$, then for any x in this set, $[x; y] = [x; 0]$ is an optimal solution to the Phase One Problem with optimal value 0 (it is feasible, with objective value 0 and no other solution can achiever lower value).

Another proof of the converse direction: if $\{x : Ax = b, \quad x \geq 0\} \neq \emptyset$, then from Farkas' lemma that the maximal value of the dual is less or equal to zero. But $\pi = 0$ is a feasible solution for the dual so that the optimal value of the dual is zero.

6.

(a) We write the dual of the problem as

$$\begin{array}{rll} \text{minimize} & \sum_i p_i + \sum_j q_j & \\ \text{subject to} & p_i + q_j \geq s_{ij}, & i, j \in \{1, 2, 3\} \\ & p, q \text{ free} & \end{array}$$

To show that there exists i and j for which $p_i + q_j \geq s_{ij}$, it is enough to show that the primal has a solution. To do this, it is enough to show that the primal's objective function is bounded. Note that since the sum of x_{ij} over i and(or) j is 1 according to primal constraints, $\sum_i \sum_j s_{ij}$ is an upper bound for the objective function. Therefore, primal has a solution and this means that the dual problem is feasible. That is, there exists i and j for which $p_i + q_j \geq s_{ij}$.

If in an optimal assignment activity i is assigned to parcel j, we have $x_{ij} = 1$. By complementary slackness, $p_i + q_j = s_{ij}$.

(b) By part (a), we have $p_i + q_j = s_{ij}$ and $p_i + q_{j'} \geq s_{ij'}$. Hence, $s_{ij} - q_j = p_i \geq s_{ij'} - q_{j'}$.

s_{ij} is the value created by locating activity i at parcel j, and q_j is the price of land j. Their difference is the net profit generated by locating activity i at parcel j.

Therefore, choosing j such that

$$s_{ij} - q_j \geq s_{ij'} - q_{j'}$$

is to choose the location for activity i with the maximum net profit.

The equilibrium in free competition achieves both primal and dual optimality. Primal objective value (where the central authority maximizes its total revenue) is equal to the dual objective value (where the individual activities minimize their total price/cost).

(c) Easiest Proof: Consider change the constraints $\sum_i x_{ij} = 1, \sum_j x_{ij} = 1$ to $\sum_i x_{ij} \leq 1, \sum_j x_{ij} \leq 1$ in the primal.

Then equality and inequality are equivalent if $s_{ij} > 0$. In the latter case, the dual variables are non-negative.

Another Proof:

Assume $\exists i, p_i < 0$. since $p_i + q_j \geq s_{ij} \forall i, j \in \{1, 2, \ldots, n\}$ and $s_{ij} > 0$, we must have $q_j > 0, \forall j$. Let $\min_i\{p_i\} = -c$. Let $p'_i = p_i + c, \forall i$ and $q'_j = q_j - c, \forall j$. Then $p'_i \geq 0, \forall i$ and since $\min_j\{q_j\} + \min_i\{p_i\} \geq s_{ij} > 0$, $q'_j > 0, \forall j$.

We still have $p'_i + q'_j = p_i + q_j \geq s_{ij}$. Therefore, we get a new feasible dual solution which gives the same objective value as before. Namely, whenever we have a negative price, we can construct an equivalent nonnegative price. Therefore, the prices can all be assumed to be nonnegative.

Problem 4-8 L&Y. To avoid confusion, we use $\mathcal{A}$ to denote the payoff matrix. In this problem, A, B are numbers.

(a) The LP can be rewritten as

$$\begin{array}{ll} \text{maximize} & min_j\{x^T a_{\cdot j}\} \\ \text{subject to} & e^T x = 1 \\ & x \geq 0 \end{array}$$

Since $e^T y = 1, y \geq 0$, we have $x^T \mathcal{A} y \geq min_j\{x^T a_{\cdot j}\} \geq A$. Hence X is guaranteed a payoff of at least A.

(b) Primal:

$$\begin{aligned} \max \quad & A + 0^T x \\ \text{subject to} \quad & 0A + e_m^T x = 1 \\ & Ae_n - \mathcal{A}^T x \le 0 \\ & x \ge 0 \end{aligned}$$

Let B,y be the dual variables corresponding to constraints $0A + e_m^T x = 1$ and $Ae_n - \mathcal{A}^T x \le 0$ respectively.

Then the dual is

$$\begin{aligned} \min \quad & B \\ \text{subject to} \quad & e_n^T y = 1 \\ & e_m B - \mathcal{A} y \ge 0 \\ & y \ge 0, B \text{ free} \end{aligned}$$

It is equivalent to the LP given in part (b).

(c) Since primal and dual are both feasible, the optimal solution exists. By strong duality theorem, $\max A = \min B$.

(d)The payoff matrix in the matching game is

$$\mathcal{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The value of this game is 0 and the optimal strategy for both X and Y is $(0.5, 0.5)$.

(e) In this game, the payoff matrix is

$$\mathcal{A} = \begin{pmatrix} 0 & 3 & -1 \\ -3 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}$$

The value of this game is 0 and the optimal strategy for both players is $(3/7, 1/7, 3/7)$.

7. Consider a linear program (P) of the form

$$\begin{aligned} \text{minimize} \quad & q^T z \\ \text{subject to} \quad & Mz \ge -q \\ & z \ge 0 \end{aligned}$$

in which the matrix M is skew symmetric; that is, $M = -M^T$.

(a) The dual problem can be written

$$\begin{array}{ll} \text{maximize} & -q^T w \\ \text{subject to} & w^T M \le q^T \\ & w \ge 0 \end{array}$$

Changing the maximization to minimization, using the skew-symmetry, and eliminating minus signs where possible enables us to rewrite the dual as the LP which is exactly the primal.

(b) The interpretation is to obtain feasible solutions to a symmetric dual pair of linear programming problems such that the *reverse* of the weak duality inequality holds. This would make the feasible solutions optimal for their respective problems.

(c) If a self-dual problem has an optimal solution, it obviously has a feasible solution. Conversely, if such a problem is feasible, then so is its dual (by part (a)). Hence the problems (which are really the same by part (b)) must have an optimal solution (by the Existence Theorem mentioned above).

8.

(a) Let $z = \|Ax - b\|_\infty$. The problem can be written as

$$\begin{array}{ll} \min & z \\ \text{subject to} & Ax + ze \ge b \\ & -Ax + ze \ge -b \\ & z \ge 0, x \text{ free} \end{array}$$

The dual of the above LP is

$$\begin{array}{ll} \max & b^T u - b^T w \\ \text{subject to} & A^T u - A^T w = 0 \\ & e^T u + e^T w \le 1 \\ & u, w \ge 0 \end{array}$$

For any vector p, let $s_i = p_i^+$ and $t_i = |p_i^-|$ for any i. Then $p_i = s_i - t_i$ and $|p_i| = s_i + t_i, \forall i$. Since p satisfies $\|p\|_1 = e^T s + e^T t \le 1$ and $A^T p = A^T s - A^T t = 0$,

s, t is a feasible solution of the dual problem. By weak duality, the optimal value of the dual problem is no more than v.

Therefore, $b^T p = b^T(s-t) \leq v$.

(b) Denote the optimal cost to the problem in part (b) as v'. From (a), we obtain $v' \leq v$. Next we will prove $v' \geq v$.

If $v = 0$. $p = 0$ is a feasible solution and the cost is $b^T p = 0$. So $v' \geq v$.

If $v \neq 0$. $\forall i$, at least one of $(Ax + ze)_i = b_i$ and $(-Ax + ze)_i = -b_i$ doesn't hold. By complementary slackness theorem, if (u^*, w^*) is dual optimal, we must have $u_i^* w_i^* = 0, \forall i$. Therefore, $u^* + w^* = |u^* - w^*|$. Let $q = u^* - w^*$. q is a feasible solution to the problem in part (b). By strong duality theorem, $b^T q = b^T(u^* - w^*) = v$. v' is the optimal value to the problem in part (b), therefore $v' \geq b^T q \geq v$.

Hence, $v' = v$.

9. Consider the feasible region of a standard LP $\{Ax = b, x \geq 0\}$, where $A \in R^{m \times n}$ is full row rank ($m \leq n$), $x \in R^n$. Suppose x is a BFS with $A_B x_B = b$, $x_N = 0$, where B is the set of basic variable indices, and N is the set of non-basic variable indices. Assume the contrary that x is not an extreme point of the feasible region, then there exist two feasible solutions $y, z \neq x$ such that $x = (y+z)/2$. This implies $y_N + z_N = 2x_N = 0$; combing with $y_N, z_N \geq 0$, we have $y_N = z_N = 0$. Then $b = Ay = A_B y_B + A_N y_N = A_B y_B$, which implies $y_B = A_B^{-1} b = x_B$. Therefore $y = x$, a contradiction.