MS&E 310 **Homework #2 Solution**
Linear Programming **October 20, 2017**

HOMEWORK ASSIGNMENT 2 SOLUTION

1. Farkas' lemma can be used to derive many other (named) theorems of the alternative. This problem concerns a few of these pairs of systems. Using Farkas's lemma, prove each of the following results.

(a) Gordan's Theorem. Exactly one of the following systems has a solution:

$$\begin{array}{rl} \text{(i)} & Ax > 0 \\ \text{(ii)} & y^T A = 0, \quad y \geq 0, \quad y \neq 0. \end{array}$$

Let b be any fixed positive vector (e.g. the all-one vector). Then, (i) has a solution iff $Ax \geq b$ has a solution: in fact, if $Ax > 0$, then one can always scale x so that $Ax \geq b$. We can write $Ax \geq b$ as:

$$Ax' - Ax'' - z = b, \; (x'; x''; z) \geq 0$$

By Farkas' lemma, if it has no solution, then we must have an y such that:

$$y^T(A, \; -A, \; -I) \leq 0, \; y^T b = 1$$

Then, y satisfies (ii). Conversely, if (ii) has no solution, then the above system has no solution, and thus $Ax \geq b$ has a solution.

(b) Stiemke's Theorem. Exactly one of the following systems has a solution:

$$\begin{array}{rl} \text{(i)} & Ax \geq 0, \quad Ax \neq 0 \\ \text{(ii)} & y^T A = 0, \quad y > 0 \end{array}$$

Let b be any fixed positive vector (e.g. the all-one vector). Then, (i) is equivalent to $Ax \geq 0$, $b^T Ax = 1$ and it can be written as:

$$\begin{pmatrix} A & -A & -I \\ b^T A & -b^T A & 0 \end{pmatrix} (x'; x''; z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; (x'; x''; z) \geq 0 \tag{1}$$

By Farkas' lemma, if it has no solution, then we must have a pair $(y';\tau)$ such that:

$$(y';\tau)^T \begin{pmatrix} A & -A & -I \\ b^T A & -b^T A & 0 \end{pmatrix} \le 0,\ (y';\tau)^T(0;1) = 1 \qquad (2)$$

Let $y = y' + \tau \cdot b$. Then, y satisfies (ii). Conversely, if (ii) has no solution, then $y^T A = 0, y \ge b$ has no solution, which means $y^T A = 0, y = y' + b, y' \ge 0$ has no solution. Thus (2) has no solution; by Farkas' lemma, (1) has a solution, thus (i) has a solution.

(c) Gale's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad Ax \le b$$
$$\text{(ii)} \quad y^T A = 0, \quad y^T b < 0, \quad y \ge 0$$

Note that (i) can be written as:

$$Ax' - Ax'' + z = b,\ (x';x'';z) \ge 0$$

By Farkas' lemma, if it has no solution, then we must have an y such that:

$$y^T(A,\ -A,\ I) \le 0,\ y^T b = 1$$

Then, $-y$ satisfies (ii). The other direction is similar.

2. Given that the dual of a linear program

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{array}$$

in standard form is

$$\begin{array}{ll} \text{maximize} & \mathbf{y}^T\mathbf{b} \\ \text{subject to} & \mathbf{y}^T A \le \mathbf{c}^T \\ & (\mathbf{y} \text{ free}) \end{array},$$

develop an appropriate dual for each of the following LPs:

(a)

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{array}$$

Solutions:

$$\begin{array}{ll} \textit{minimize} & y^T b \\ \textit{subject to} & y^T A \geq c^T \\ & y \textit{ free.} \end{array}$$

(b)

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Solutions:

$$\begin{array}{ll} \textit{maxmize} & y^T b \\ \textit{subject to} & y^T A \leq c^T \\ & y \geq 0. \end{array}$$

(c)

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \bar{A}\mathbf{x} \geq \bar{\mathbf{b}} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Solutions:

$$\begin{array}{ll} \textit{maxmize} & y^T b + s^T \bar{b} \\ \textit{subject to} & y^T A + s^T \bar{A} \leq c^T \\ & y \textit{ free} \\ & s \geq 0. \end{array}$$

3. Consider the auction problem in Lecture note #4. The LP pricing problem has an objective

$$\pi^T\mathbf{x} - z$$

where the scalar

$$z = \max[A\mathbf{x}]$$

is the maximum number of contracts among all states (recall that $A\mathbf{x} \in R^m$ is a vector representing the number of contracts in each state). Thus, z represents the *worst-case* payback amount. Now assuming that the auction organizer knows the discrete probability distribution, say $\mathbf{v} \in R^m_+$, for each state to win. Then the *expected* payback amount would be

$$\left(\sum_{i=1}^{n} v_i \cdot [Ax]_i\right) = \mathbf{v}^T A\mathbf{x}$$

Develop an LP model to decide the contract award vector $\mathbf{x}$ and to price each state using the expected payback rather than the worst-case payback, that is, using the objective function

$$\pi^T\mathbf{x} - \mathbf{v}^T A\mathbf{x}$$

in the LP setting. How to solve the problem faster? Moreover, explain the price properties using duality and/or complementarity.

Solutions: The LP model can be written as

$$\begin{array}{ll} \textit{maxmize} & \pi^T x - v^T z \\ \textit{subject to} & Ax - z \le 0 \\ & x \le \bar{v} \\ & (x, z) \ge 0. \end{array}$$

where decision variables $x \in R^n$ and $z \in R^m$, and $\bar{v}$ is the quantity limit.

The dual of the problem is

$$\begin{array}{lrl} \min & \bar{v}^T y & \\ s.t. & A^T p + y & \ge \pi, \\ & p & \le v, \\ & (p, y) & \ge 0. \end{array}$$

The complementarity conditions imply that Most of these conditions are the same

$x_j > 0$	$a_j^T p + y_j = \pi_j$ so that $a_j^T p \le \pi_j$
$0 < x_j < \bar{v}_j$	$y_j = 0$ as well so that $a_j^T p = \pi_j$
$x_j = 0$	$y_j = 0$ so that $a_j^T p \ge \pi_j$
$z_i > 0$	$p_j = v_j$
$z_i = 0$	$p_j \le v_j$

as those in our earlier model. One interesting case is when $z_i = 0$, that is, nobody bid on state i, the price for that state can be any number between 0 and v_i. One simple case is let $p = v$ and we still have $e^T p = 1$.

4. Strict Complementarity Theorem: Consider the LP problem

$$\begin{array}{lll} (LP) & \text{maximize} & \mathbf{c}^T \mathbf{x} = \sum_{j=1}^n c_j x_j \\ & \text{subject to} & \sum_{j=1}^n \mathbf{a}_j x_j = A\mathbf{x} \le \mathbf{b},\ \mathbf{0} \le \mathbf{x} \le \mathbf{e}; \end{array}$$

where data $A \in R^{m \times n}$, $\mathbf{a}_j \in R^m$, $\mathbf{c} \in R^n$, $\mathbf{b} \in R^m$ and $\mathbf{e}$ is the vector of all ones, and variables $\mathbf{x} \in R^n$. You may interpret this is a linear program to sell the items of inventory $\mathbf{b}$ to n customers such that the revenue is maximized.

Suppose the problem is feasible and bounded.

(i) Write down the dual of the problem. What are the interpretations of the dual price vector associated with the constraints $A\mathbf{x} \le \mathbf{b}$ and the dual price vector associated with the constraints $\mathbf{x} \le \mathbf{e}$?

$$\begin{array}{lll} (LD) & \textit{minimize} & b^T y + e^T s \\ & \textit{subject to} & A^T y + s \ge c, \;\; y, s \ge 0. \end{array}$$

$y_i : i = 1, 2, \cdots, m$: *price for item i which has inventory b_i;*

$s_j : j = 1, 2, \cdots, n$: *the difference between customer j's internal cost and external revenue.*

(ii) What properties does a strictly complementary solution have for this linear program pair?

Assume x, p, s is a strictly complementary solution.

The strictly complementarity conditions imply that

$1 > x_j > 0$	$a_j^T y + s_j = c_j$ and $s_j = 0$ so that $a_j^T y = c_j$
$x_j = 0$	$a_j^T y + s_j > c_j$ and $s_j = 0$ so that $a_j^T y > c_j$
$x_j = 1$	$a_j^T y + s_j = c_j$ and $s_j > 0$ so that $a_j^T y < c_j$

(iii) Suppose the linear program pair has a strictly complementary primal solution $\mathbf{x}^*$ such that $x_j^* = 0$ or $x_j^* = 1$ for all j, and let $\mathbf{y}^*$ be a strictly complementary dual price vector associated with the constraints $A\mathbf{x} \le \mathbf{b}$. Now consider a on-line linear program where customer $(c_j, \mathbf{a}_j)$ comes sequentially, and the seller have to make a decision $x_j = 0$ or $x_j = 1$ as soon as the customer arrives. Prove that the following mechanism or decision rule, given $\mathbf{y}^*$ being known, is optimal: If $c_j > \mathbf{a}_j^T \mathbf{y}^*$ then set $x_j = 1$; otherwise, set $x_j = 0$.

Since the linear program pair has a strictly complementary primal solution x^* such that $x_j^* = 0$ or $x_j^* = 1$ for all j. The correctness of the mechanism follows directly from part (b).

5. Consider a system of m linear equations in n nonnegative variables, say

$$Ax = b, \quad x \ge 0.$$

Assume the right-hand side vector b is nonnegative. Now consider the (related) linear program

$$\begin{array}{ll} \text{minimize} & e^T y \\ \text{subject to} & Ax + Iy = b \\ & x \ge 0, \;\; y \ge 0 \end{array}$$

where e is the m-vector of all ones, and I is the $m \times m$ identity matrix. This linear program is called a *Phase One Problem.*

(a) Write the dual of the Phase One Problem.

$$\begin{aligned} \text{maximize} \quad & b^T \pi \\ \text{subject to} \quad & A^T \pi \leq 0 \\ & \pi \leq e \end{aligned}$$

(b) Show that the Phase One Problem always has a basic feasible solution.

Obviously $[x; y] = [0; b]$ *is a basic solution to the Phase One Problem; since* b *is nonnegative by the assumption, it is also a feasible solution.*

(c) Using theorems proved in class, show that the Phase One Problem always has an optimal solution.

Since the Phase I problem is feasible, and its objective value is bounded from below by 0 *(the dual of Phase I has a feasible solution* $\pi = 0$*).*

(d) Write the complementary slackness conditions for the Phase One Problem.

$$\begin{aligned} x_j(-A^T\pi)_j &= 0 \ \forall \ j = 1, ..., n \\ y_i(1 - \pi_i) &= 0 \ \forall \ i = 1, ..., m. \end{aligned}$$

(e) Prove that $\{x : Ax = b, \quad x \geq 0\} \neq \emptyset$ if and only if the optimal value of the objective function in the corresponding Phase One Problem is zero.

If the optimal value of the Phase one problem is zero, then we must have also the optimal solution $(x \geq 0, y = 0)$ *and that* $Ax = b$*, that is,* $\{x : Ax = b, \quad x \geq 0\} \neq \emptyset$*. Conversely, if* $\{x : Ax = b, \quad x \geq 0\} \neq \emptyset$*, then for any* x *in this set,* $[x; y] = [x; 0]$ *is an optimal solution to the Phase One Problem with optimal value* 0 *(it is feasible, with objective value* 0 *and no other solution can achieve a lower value).*

Another proof of the converse direction: if $\{x : Ax = b, \quad x \geq 0\} \neq \emptyset$*, then from Farkas' lemma that the maximal value of the dual is less or equal to zero. But* $\pi = 0$ *is a feasible solution for the dual so that the optimal value of the dual is zero.*

6. Exercise 4.9-7

(a) We write the dual of the problem as

$$\begin{array}{rll} \text{minimize} & \sum_i p_i + \sum_j q_j & \\ \text{subject to} & p_i + q_j \geq s_{ij}, & i,j \in \{1,...,n\} \\ & p, q \text{ free} & \end{array}$$

To show that there exists p and q for which $p_i + q_j \geq s_{ij}$, it suffices to show that the primal is feasible and bounded. One feasible solution to the primal is $x_{ii} = 1$ for $i = 1,...,n$, and $x_{ij} = 0$ for $i \neq j$. Note that since the sum of x_{ij} over i (or j) is 1 according to primal constraints, $\sum_i \sum_j s_{ij}$ is an upper bound for the objective function.

If in an optimal assignment activity i is assigned to parcel j, we have $x_{ij} = 1$. By complementary slackness, $p_i + q_j = s_{ij}$.

(b) By part (a), we have $p_i + q_j = s_{ij}$ and $p_i + q_{j'} \geq s_{ij'}$. Hence, $s_{ij} - q_j = p_i \geq s_{ij'} - q_{j'}$.

s_{ij} is the value created by locating activity i at parcel j, and q_j is the price of land j. Their difference is the net profit generated by locating activity i at parcel j.

Therefore, choosing j such that

$$s_{ij} - q_j \geq s_{ij'} - q_{j'}$$

is to choose the location for activity i with the maximum net profit.

The equilibrium in free competition achieves both primal and dual optimality. Primal objective value (where the central authority maximizes its total revenue) is equal to the dual objective value (where the individual activities minimize their total price/cost).

(c) Easiest Proof: Consider change the constraints $\sum_i x_{ij} = 1, \sum_j x_{ij} = 1$ to $\sum_i x_{ij} \leq 1, \sum_j x_{ij} \leq 1$ in the primal.

Then equality and inequality are equivalent if $s_{ij} > 0$. In the latter case, the dual variables are non-negative.

Another Proof:

Assume $\exists i, p_i < 0$. since $p_i + q_j \geq s_{ij} \forall i, j \in \{1, 2, \ldots, n\}$ and $s_{ij} > 0$, we must have $q_j > 0, \forall j$. Let $\min_i\{p_i\} = -c$. Let $p'_i = p_i + c, \forall i$ and $q'_j = q_j - c, \forall j$. Then $p'_i \geq 0, \forall i$ and since $\min_j\{q_j\} + \min_i\{p_i\} \geq s_{ij} > 0$, $q'_j > 0, \forall j$.

We still have $p'_i + q'_j = p_i + q_j \geq s_{ij}$. Therefore, we get a new feasible dual solution which gives the same objective value as before. Namely, whenever we have a negative price, we can construct an equivalent nonnegative price. Therefore, the prices can all be assumed to be nonnegative.

7. Exercise 4.9-10

Consider the primal linear program in the standard form. Suppose that this program and its dual are feasible. Let y be a known optimal solution to the dual.

(a) If the k-th equation of the primal is multiplied by $\mu \neq 0$, determine an optimal solution w to the dual of this new problem.

Multiplying the k-th equality constraint of the primal by μ doesn't change the primal feasible region, thus doesn't change the primal optimal solution. Hence the optimal objective values of primal and dual problems remain the same.

Let $A'x = b'$ denote the new equality constraint. Construct w such that $w_k = y_k/\mu$ and $w_i = y_i$ for $i \neq k$. Then it can be easily verified that $w^T A' \leq c^T$, thus w is feasible to the new dual problem. Also $w^T b' = y^T b$, so w achieves the optimal objective value, and it is the optimal solution.

(b) Suppose that, in the original primal, we add μ times the k-th equation to the r-th equation. What is an optimal solution w to the corresponding dual problem?

Similar to part (a), the optimal objective value does not change. Let $w_k = y_k - \mu y_r$, and $w_i = y_i$ for $i \neq k$, then w is the optimal solution to the new dual problem.

(c) Suppose, in the original primal, we add μ times the k-th row of A to c. What is an optimal solution to the corresponding dual problem?

$w = y + \mu e_k$, where e_k is the unit vector with the k-th entry being 1.

8. Let A be an m by n matrix and let $\mathbf{b}$ be a vector in R^m. We consider the problem of minimizing $\|A\mathbf{x} - \mathbf{b}\|_\infty$ over all $\mathbf{x} \in R^n$. Let v be the value of the optimal cost.

(a) Let $\mathbf{p}$ be any vector in R^m that satisfies $\|\mathbf{p}\|_1 = \sum_{i=1}^m |p_i| \le 1$ and $A^T\mathbf{p} = \mathbf{0}$. Show that $\mathbf{b}^T\mathbf{p} \le v$.

Let $z = \|Ax - b\|_\infty$. The problem can be written as

$$\begin{array}{ll} \textit{min} & z \\ \textit{subject to} & Ax + ze \ge b \\ & -Ax + ze \ge -b \\ & z \ge 0, x \textit{ free} \end{array}$$

The dual of the above LP is

$$\begin{array}{ll} \textit{max} & b^T u - b^T w \\ \textit{subject to} & A^T u - A^T w = 0 \\ & e^T u + e^T w \le 1 \\ & u, w \ge 0 \end{array}$$

For any vector p, let $s_i = p_i^+$ and $t_i = |p_i^-|$ for any i. Then $p_i = s_i - t_i$ and $|p_i| = s_i + t_i$, $\forall i$. Since p satisfies $\|p\|_1 = e^T s + e^T t \le 1$ and $A^T p = A^T s - A^T t = 0$, s, t is a feasible solution of the dual problem. By weak duality, the optimal value of the dual problem is no more than v.

Therefore, $b^T p = b^T(s - t) \le v$.

(b) In order to obtain the best possible lower bound of the form considered in part (a), we form the linear programming problem

$$\begin{array}{ll} \text{maximize} & \mathbf{b}^T\mathbf{p} \\ \text{subject to} & A^T\mathbf{p} = \mathbf{0} \\ & \|\mathbf{p}\|_1 \le 1. \end{array}$$

Show that the optimal cost on this problem is equal to v.

Denote the optimal cost to the problem in part (b) as v'. From (a), we obtain $v' \le v$. Next we will prove $v' \ge v$.

If $v = 0$. $p = 0$ is a feasible solution and the cost is $b^T p = 0$. So $v' \geq v$.

If $v \neq 0$. $\forall i$, at least one of $(Ax + ze)_i = b_i$ and $(-Ax + ze)_i = -b_i$ doesn't hold. By complementary slackness theorem, if (u^*, w^*) is dual optimal, we must have $u_i^* w_i^* = 0, \forall i$. Therefore, $u^* + w^* = |u^* - w^*|$. Let $q = u^* - w^*$. q is a feasible solution to the problem in part (b). By strong duality theorem, $b^T q = b^T(u^* - w^*) = v$. v' is the optimal value to the problem in part (b), therefore $v' \geq b^T q \geq v$.

Hence, $v' = v$.

9. Prove that BFS is an extreme point of the feasible region in the LP standard form.

Consider the feasible region of a standard LP $\{Ax = b, x \geq 0\}$, where $A \in R^{m \times n}$ is full row rank ($m \leq n$), $x \in R^n$. Suppose x is a BFS with $A_B x_B = b$, $x_N = 0$, where B is the set of basic variable indices, and N is the set of non-basic variable indices. Assume the contrary that x is not an extreme point of the feasible region, then there exist two feasible solutions $y, z \neq x$ such that $x = (y + z)/2$. This implies $y_N + z_N = 2x_N = 0$; combing with $y_N, z_N \geq 0$, we have $y_N = z_N = 0$. Then $b = Ay = A_B y_B + A_N y_N = A_B y_B$, which implies $y_B = A_B^{-1} b = x_B$. Therefore $y = x$, a contradiction.

10. (Lemma 2 on slide 25.) The discounted MDP primal LP is given by

$$\begin{array}{ll} \text{minimize} & \sum_{i \in S} \sum_{a \in A} c(i, a) x(i, a) \\ \text{subject to} & \sum_{a \in A} x(j, a) = 1 + \gamma \sum_{i \in S} \sum_{a \in A} x(i, a) p_{ij}(a), \forall j \in S \\ & x(i, a) \geq 0, \forall i \in S, a \in A. \end{array}$$

where S denotes the state space, A the action space, $c(i, a)$ the cost function of choosing action a when the state is x, $p_{ij}(a)$ the probability of transitioning from state i to j under action a. The decision variables are $x(i, a)$, for $i \in S, a \in A$.

Show that it has the following properties:

(a) The feasible set is bounded. More precisely, for every feasible $x \geq 0$, $e^T x = \frac{m}{1-\gamma}$.

Adding up all the equality constraints yields $e^T x = \frac{m}{1-\gamma}$. Since a feasible solution x is non-negative, we have $0 \leq x \leq \frac{m}{1-\gamma}$ bounded.

(b) There is a one-to-one correspondence between a stationary policy of the original discounted MDP and a basic feasible solution (BFS) of the primal.

First we show a BFS x corresponds to a stationary policy. Let B denote the index set of basic variables, then $|B| = m$. Suppose B does not contain any state-action *pair for a certain state k, then the k-th equality constraint fails to hold:*

$$\sum_{a \in A} x(k,a) = 0 \neq 1 + \gamma \sum_{i \in S} \sum_{a \in A} x(i,a) p_{ik}(a) \geq 1.$$

Therefore B contains exactly one state-action pair for each state, and corresponds to a stationary policy of the discounted MDP problem.

Next we show that a stationary policy π corresponds to a BFS. Suppose $\pi(i) = a_i$ for $i = 1, 2, ..., m$. Let $B = \{(i, a_i) | i = 1, 2, ..., m, a_i \in A\}$ be an index set. Then selecting the corresponding columns of the equality constraint matrix yields $A_B = I - \gamma P_B$, where $P_B(i,j) = p_{ji}(a_j)$. And x_B is the solution to $A_B x_B = e$. Observe that the diagonal entries of A_B are positive, and the off-diagonal entries are non-positive, thus A_B is of full rank. Therefore A_B is a basis, and x such that $x_B = A_B^{-1} e$ and $x_N = 0$ is a basic solution. It remains to show that x is feasible, that is, $x_B \geq 0$. Suppose not, then the system $\{A_B x_B = e, x_B \geq 0\}$ is infeasible. By Farka's Lemma, there exists y such that $y^T A_B \leq 0$ and $y^T e > 0$. Wlog, assume y_1 is the maximum entry in y. Then $y^T e > 0$ implies that $y_1 > 0$. Also,

$$0 \geq (y^T A_B)_1 = (y^T (I - \gamma P_B))_1 = y_1 - \gamma y^T p_1,$$

where p_1 denote the first column of P_B. Since p_1 is a probability vector, we have $y^T p_1 \leq y_1$, then $y_1 - \gamma y^T p_1 \geq (1 - \gamma) y_1 > 0$ yields a contradiction.

(c) Every policy or BFS basis has the Leontief substitution form $A_B = I - \gamma P_B$.

Immediately follows from part (b).

(d) Let x^π be a basic feasible solution. Then any basic variable, say x_j^π, has its value $1 \leq x_j^\pi \leq \frac{m}{1-\gamma}$.

x^π is feasible, thus by part (a) we know $x_j^\pi \leq \frac{m}{1-\gamma}$. It remains to show $x_j^\pi \geq 1$, which is obvious, since the j-th equality constraint requires

$$\sum_{a \in A} x^\pi(j,a) = x_j^\pi = 1 + \gamma \sum_{i \in S} \sum_{a \in A} x^\pi(i,a) p_{ij}(a) \geq 1.$$