Dual Interpretations and Duality Applications

Yinyu Ye
Department of Management Science and Engineering
Stanford University
Stanford, CA 94305, U.S.A.

http://www.stanford.edu/~yyye

This week: Appendix B, Chapters 2.2, 2.6, 3.1-3.6, 6.3-6.4
Production Problem I

\[
\text{max } p^T x \quad \text{s.t.} \quad Ax \leq r, \quad x \geq 0
\]

where

- \( p \): profit margin vector
- \( A \): resources consumption rate matrix
- \( r \): available resource vector
- \( x \): production level decision vector
Production Problem II: Liquidation Pricing

- $y$: the fair price vector
- $A^T y \geq p$: competitiveness
- $y \geq 0$: positivity
- $\min r^T y$: minimize the total liquidation cost
maximize \[ x_1 + 2x_2 \]
subject to \[ x_1 \leq 1 \]
\[ x_2 \leq 1 \]
\[ x_1 + x_2 \leq 1.5 \]
\[ x_1, x_2 \geq 0. \]

minimize \[ y_1 + y_2 + 1.5y_3 \]
subject to \[ y_1 + y_3 \geq 1 \]
\[ y_2 + y_3 \geq 2 \]
\[ y_1, y_2, y_3 \geq 0. \]
**Optimal Value Function and Shadow Prices**

\[ z(b) = \text{minimize} \quad c^T x \]

subject to \( Ax = b, \ x \geq 0. \)

Suppose a new right-hand-vector \( b^+ \) such that

\[ b^+_k = b_k + \delta \quad \text{and} \quad b^+_i = b_i, \ \forall i \neq k. \]

Then, the optimal dual solution \( y^* \) has a property

\[ y^*_k = (z(b^+) - z(b))/\delta \]

as long as \( y^* \) remains the dual optimal solution for \( b^+ \), because

\[ z(b^+) = (b^+)^T y^* = z(b) + \delta \cdot y^*_k. \]

Thus, the optimal dual value is the rate of the net change of the optimal objective value over the net change of an entry of the right-hand-vector resources, i.e.,

\[ \nabla_z(b) = y^*. \]
Application in the Wassestein Barycenter Problem

Find distribution of $x_i, i = 1, 2, 3, 4$ to minimize

$$\min \ W D_l(x) + W D_m(x) + W D_r(x)$$

s.t. \[x_1 + x_2 + x_3 + x_4 = 9, \quad x_i \geq 0, \ i = 1, 2, 3, 4.\]

The objective is a nonlinear function, but its gradient vector $\nabla W D_l(x), \nabla W D_m(x)$ and $\nabla W D_l(x)$
are shadow prices of the three sub-transportation problems—popularly used in Hierarchy Optimization.
Recall Transportation Problem

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{j=1}^{n} x_{ij} = s_i, \quad \forall i = 1, \ldots, m \\
& \quad \sum_{i=1}^{m} x_{ij} = d_j, \quad \forall j = 1, \ldots, n \\
& \quad x_{ij} \geq 0, \quad \forall i, j.
\end{align*}
\]
SupplyDemand

C11, x11

Demand

Supply
**Transportation Dual: Economic Interpretation**

\[
\text{max} \quad \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \\
\text{s.t.} \quad u_i + v_j \leq c_{ij}, \forall i, j.
\]

\( u_i \): supply site unit price

\( v_i \): demand site unit price

\( u_i + v_j \leq c_{ij} \): competitiveness
Max-Flow and Min-Cut

Given a directed graph with nodes 1, ..., m and edges \( A \), where node 1 is called source and node m is called the sink, and each edge \((i, j)\) has a flow rate capacity \( k_{ij} \). The Max-Flow problem is to find the largest possible flow rate from source to sink.

Let \( x_{ij} \) be the flow rate from node \( i \) to node \( j \). Then the problem can be formulated as

\[
\begin{align*}
\text{maximize} & \quad x_{m1} \\
\text{subject to} & \quad \sum_{j: (j, 1) \in A} x_{j1} - \sum_{j: (1, j) \in A} x_{1j} + x_{m1} = 0, \\
& \quad \sum_{j: (j, i) \in A} x_{ji} - \sum_{j: (i, j) \in A} x_{ij} = 0, \forall i = 2, \ldots, m - 1, \\
& \quad \sum_{j: (j, m) \in A} x_{jm} - \sum_{j: (m, j) \in A} x_{mj} - x_{m1} = 0, \\
& \quad 0 \leq x_{ij} \leq k_{ij}, \forall (i, j) \in A.
\end{align*}
\]
The dual of the Max-Flow problem

minimize \[ \sum_{(i,j) \in A} k_{ij} z_{ij} \]
subject to \[ -y_i + y_j + z_{ij} \geq 0, \forall (i, j) \in A, \]
\[ y_1 - y_m = 1, \]
\[ z_{ij} \geq 0, \forall (i, j) \in A. \]

\( y_i \): node potential value. At an optimal solution has property \( y_1 = 1, y_m = 0 \) and for all other \( i \):

\[
y_i = \begin{cases} 
1 & \text{if } i \in S \\
0 & \text{if } i \notin S 
\end{cases}
\]

This problem is called the Min-Cut problem.
Recall the cost-to-go value of the reinforcement learning LP problem:

maximize \( y \)
\[
\sum_{i=1}^{m} y_i
\]
subject to
\[
y_1 - \gamma p_j^T y \leq c_j, \quad j \in A_1
\]
\[
\quad \ldots
\]
\[
y_i - \gamma p_j^T y \leq c_j, \quad j \in A_i
\]
\[
\quad \ldots
\]
\[
y_m - \gamma p_j^T y \leq c_j, \quad j \in A_m.
\]

minimize \( x \)
\[
\sum_{j \in A_1} c_j x_j + \ldots + \sum_{j \in A_m} c_j x_j
\]
subject to
\[
\sum_{j \in A_1} (e_1 - \gamma p_j) x_j + \ldots + \sum_{j \in A_m} (e_m - \gamma p_j) x_j = e,
\]
\[
\quad \ldots x_j \quad \ldots \quad \geq 0, \quad \forall j,
\]
where \( e_i \) is the unit vector with 1 at the \( i \)th position and 0 everywhere else.
Interpretation of the Dual of the RL-LP

Variable $x_j, j \in A_i$, is the state-action frequency or called flux, or the expected present value of the number of times that an individual is in state $i$ and takes state-action $j$.

Thus, solving the problem entails choosing a state-action frequencies/fluxes that minimizes the expected present value of total costs for the infinite horizon, where the RHS is $(1; 1; 1; 1; 1; 1)$:

<table>
<thead>
<tr>
<th>x:</th>
<th>(0₁)</th>
<th>(0₂)</th>
<th>(1₁)</th>
<th>(1₂)</th>
<th>(2₁)</th>
<th>(2₂)</th>
<th>(3₁)</th>
<th>(3₂)</th>
<th>(4₁)</th>
<th>(5₁)</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>c:</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(0)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(1)</td>
<td>$-\gamma$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(2)</td>
<td>0</td>
<td>$-\gamma/2$</td>
<td>$-\gamma$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(3)</td>
<td>0</td>
<td>$-\gamma/4$</td>
<td>0</td>
<td>$-\gamma/2$</td>
<td>$-\gamma$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(4)</td>
<td>0</td>
<td>$-\gamma/8$</td>
<td>0</td>
<td>$-\gamma/4$</td>
<td>0</td>
<td>$-\gamma/2$</td>
<td>$-\gamma$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(5)</td>
<td>0</td>
<td>$-\gamma/8$</td>
<td>0</td>
<td>$-\gamma/4$</td>
<td>0</td>
<td>$-\gamma/2$</td>
<td>0</td>
<td>$-\gamma$</td>
<td>$-\gamma$</td>
<td>1</td>
<td>$1-\gamma$</td>
</tr>
</tbody>
</table>

where state 5 is the absorbing state that has a infinite loops to itself.
The optimal dual solution is

\[ x^*_0 = 1, \ x^*_1 = 1 + \gamma, \ x^*_2 = 1 + \gamma + \gamma^2, \ x^*_3 = 1 + \gamma + \gamma^2 + \gamma^3, \ x^*_4 = 1, \ x^*_5 = \frac{1 + \gamma \cdot x^*_3}{1 - \gamma}. \]
Application: Combinatorial Auction Pricing I

Given the $m$ different states that are mutually exclusive and exactly one of them will be true at the maturity. A contract on a state is a paper agreement so that on maturity it is worth a notional $1$ if it is on the winning state and worth $0$ if it is not on the winning state. There are $n$ orders betting on one or a combination of states, with a price limit and a quantity limit.
Combinatorial Auction Pricing II: an order

The \( j \)th order is given as \((a_j \in R_m^+, \pi_j \in R_+, q_j \in R_+)\): \( a_j \) is the combination betting vector where each component is either 1 or 0

\[
a_j = \begin{pmatrix}
a_{1j} \\
a_{2j} \\
\vdots \\
a_{mj}
\end{pmatrix},
\]

where 1 is winning and 0 is non-winning; \( \pi_j \) is the price limit for one such a contract, and \( q_j \) is the maximum number of contracts the better like to buy.
### World Cup Information Market

<table>
<thead>
<tr>
<th>Order:</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Brazil</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Italy</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Germany</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>France</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Bidding Prize: $\pi$</td>
<td>0.75</td>
<td>0.35</td>
<td>0.4</td>
<td>0.95</td>
<td>0.75</td>
</tr>
<tr>
<td>Quantity limit: $q$</td>
<td>10</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>Order fill: $x$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$x_4$</td>
<td>$x_5$</td>
</tr>
</tbody>
</table>
Let $x_j$ be the number of contracts awarded to the $j$th order. Then, the $j$th better will pay the amount

$$\pi_j \cdot x_j$$

and the total collected amount is

$$\sum_{j=1}^{n} \pi_j \cdot x_j = \pi^T \mathbf{x}$$

If the $i$th state is the winning state, then the auction organizer need to pay back

$$\left( \sum_{j=1}^{n} a_{ij} x_j \right)$$

The question is, how to decide $\mathbf{x} \in \mathbb{R}^n$. 
Combinatorial Auction Pricing IV: LP model

\[ \begin{align*}
\max \quad & \pi^T x - z \\
\text{s.t.} \quad & Ax - e \cdot z \leq 0, \\
\quad & x \leq q, \\
\quad & x \geq 0.
\end{align*} \]

\( \pi^T x \): the optimistic amount can be collected. \( z \): the worst-case amount need to pay back.
Combinatorial Auction V: The dual

\[
\begin{align*}
\min & \quad q^T y \\
\text{s.t.} & \quad A^T p + y \geq \pi, \\
& \quad e^T p = 1, \\
& \quad (p, y) \geq 0.
\end{align*}
\]

\(p\) represents the state price.

What is \(y\)?

Price information gaps/differentials/slacks where their weighted sum we like to minimize.
Combinatorial Auction V: Strict Complementarity

<table>
<thead>
<tr>
<th>$x_j &gt; 0$</th>
<th>$a_j^T p + y_j = \pi_j$ and $y_j \geq 0$ so that $a_j^T p \leq \pi_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; x_j &lt; q_j$</td>
<td>$y_j = 0$ so that $a_j^T p = \pi_j$</td>
</tr>
<tr>
<td>$x_j = q_j$</td>
<td>$y_j &gt; 0$ so that $a_j^T p &lt; \pi_j$</td>
</tr>
<tr>
<td>$x_j = 0$</td>
<td>$a_j^T p + y_j &gt; \pi_j$ and $y_j = 0$ so that $a_j^T p &gt; \pi_j$</td>
</tr>
</tbody>
</table>

The price is **Fair**: 

$$p^T(Ax - e \cdot z) = 0 \quad \text{implies} \quad p^TAx = p^T e \cdot z = z;$$

that is, the worst case cost equals the worth of total shares. Moreover, if a lower bid wins the auction, so does the higher bid on any same type of bids.
### World Cup Information Market Result

<table>
<thead>
<tr>
<th>Order:</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
<th>State Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>Brazil</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.35</td>
</tr>
<tr>
<td>Italy</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>Germany</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>France</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Bidding Price:π</td>
<td>0.75</td>
<td>0.35</td>
<td>0.4</td>
<td>0.95</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>Quantity limit:q</td>
<td>10</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Order fill:x*</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

**Question 1:** The uniqueness of dual prices?
Combinatorial Auction Pricing VI: convex programming model

\[
\begin{align*}
\max & \quad \pi^T x - z + u(s) \\
\text{s.t.} & \quad Ax - e \cdot z + s = 0, \\
& \quad x \leq q, \\
& \quad x, s \geq 0.
\end{align*}
\]

\(u(s)\): a value function for the market organizer on slack shares.

If \(u(\cdot)\) is a strictly concave function, then the state price vector is unique.

Question 2: Online allocation?