Midterm

MS&E 310

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Question 1 (30pts)

This problem is concerned with using an optimality criterion for linear programming to decide whether the vector $\boldsymbol{x}^* = (0, 2, 0, 7, 0)$ is optimal for the linear program:

$$
\begin{aligned}
\text{(P)}\ \ \max \quad & 8x_1 - 9x_2 + 12x_3 + 4x_4 + 11x_5, \\
\text{subject to} \quad & 2x_1 - 3x_2 + 4x_3 + x_4 + 3x_5 \leq 1, \\
& x_1 + 7x_2 + 3x_3 - 2x_4 + x_5 \leq 1 \\
& 5x_1 + 4x_2 - 6x_3 + 2x_4 + 3x_5 \leq 22, \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

(6 pts each)

(a) Write down the dual (D) of the problem (P)

Answer

The dual problem (D) is:

$$
\begin{aligned}
\min \quad & y_1 + y_2 + 22y_3 \\
\text{subject to} \quad & 2y_1 + y_2 + 5y_3 \geq 8 \\
& -3y_1 + 7y_2 + 4y_3 \geq -9 \\
& 4y_1 + 3y_2 - 6y_3 \geq 12 \\
& y_1 - 2y_2 + 2y_3 \geq 4 \\
& 3y_1 + y_2 + 3y_3 \geq 11 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

(b) Noting that the constraints of (P) have the form $\boldsymbol{Ax} \leq \boldsymbol{b}, \boldsymbol{x} \geq 0$, write down the support of the vectors $\boldsymbol{x}^*$ and $\boldsymbol{b} - \boldsymbol{Ax}^*$.

Answer

$\text{support}(\boldsymbol{x}^*) = \{2, 4\}$, $\text{support}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^*) = \{2\}$

(c) For the moment, assume that $\boldsymbol{x}^*$ is an optimal solution to (P). Write down a system of equations in the dual variables that must hold as a consequence of this assumption.

Answer

From part (b), we find that the second and fourth dual constraints must hold as equations by complementarity. Likewise, the second dual variable must be zero. Thus, we have:

$$\begin{aligned} -3y_1 + 7y_2 + 4y_3 &= -9 \\ y_1 - 2y_2 + 2y_3 &= 4 \\ y_2 &= 0 \end{aligned}$$

(d) Solve the system in part (c).

Answer

After solving the system in part (c), we see that: $(y_1, y_2, y_3) = (17/5, 0, 3/10)$ assuming that $\boldsymbol{x}^*$ is an optimal solution.

(e) Decide whether $\boldsymbol{x}^*$ is actually an optimal solution to (P).

Answer

Although the objective values of both the primal and the dual are equal, the solution obtained from part (d) is not feasible for (D). Thus, $\boldsymbol{x}^*$ is not optimal for (P).

Question 2 (25 pts)

In this question, we will develop linear programming approximation for an integer optimization problem. Consider a function

$$\phi(\mathbf{X}) = \boldsymbol{\alpha}^T\mathbf{X} + \mathbf{X^T W X} = \sum_i \alpha_i X_i + \sum_{i \neq j} w_{ij} X_i X_j,$$

where $\mathbf{X} = (X_1, X_2, ..., X_n)$ is a binary vector ($X_i \in \{0, 1\}$) and $\boldsymbol{W}_{ii} = 0$. Our goal is to

$$\text{(D-OPT)} \ \max_{\mathbf{X}} \quad \phi(\mathbf{X})$$
$$\text{subject to } X_i \in \{0, 1\}$$

This is a simplified version of the Ising model's potential function widely used statistical physics and computer vision. You may think of the function value $\phi(\mathbf{X})$ proportional to the log likelihood of a configuration of atomic spinnings or image pixel values. To maximize the objective function is actually the procedure to find the most probable configuration. Since this integer programming problem is pretty hard to solve, people always try to find an LP approximation.

(a) (15 pts) Formulate an LP approximation for the above D-OPT problem by using the technique of relaxation. (Hint. Consider relaxing $X_i \in \{0, 1\}$ into $X_i \in [0, 1]$. And also you should notice that there are quadratic terms that need to be removed. You may want to introduce $Y_{ij} = X_i X_j$ and investigate the relation between X_i, X_j and Y_{ij}.)

Answer

The following LP could be an approximation for the original D-OPT problem:

$$\begin{aligned} \max_{\boldsymbol{x},\boldsymbol{y}} \quad & \sum_i \alpha_i x_i + \sum_{i \neq j} w_{ij} y_{ij}, \\ \text{subject to} \quad & 0 \leq x_i \leq 1, \quad 0 \leq y_{ij} \leq 1, \\ & y_{ij} + 1 - x_i - x_j \geq 0, \\ & y_{ij} - x_i \leq 0, \end{aligned}$$

$$y_{ij} - x_j \leq 0.$$

The second line is a relaxation for discrete constraints and the third line is a relaxation for the constraints $y_{i,j} = x_i x_j$.

(b) (10 pts) What is the relation of the optimal values of the D-OPT, your LP, and its dual? We don't require you to write out the dual explicitly here but you may do that if you are interested.

Answer

For all the optimal solutions of the D-OPT, they are all feasible for the LP problem. So the optimal value of LP would an upper bound for the D-OPT problem. Since the primal problem is both feasible and bounded, the strong duality holds so that the LP and its dual have same optimal value. Therefore, for the optimal values, D-OPT $\leq$ LP = Dual LP.

Question 3 (25 pts)

Quantile regression is a type of regression model in statistics and econometrics. While the least square regression models the expectation of target variable Y, the quantile regression models its quantile. In this question, we will investigate the computational aspect of the quantile regression. The model assumes the τ-quantile of the target random variable $Y \in R$ is a linear function of the response variable

$$Q_Y(\tau) = \boldsymbol{\beta}^T \boldsymbol{x},$$

where $\tau \in (0,1)$, $\boldsymbol{\beta} \in R^p$ and $\boldsymbol{x} \in R^p$. Given the response data matrix $\boldsymbol{X} \in R^{n\times p}$ $(n > p)$ and the target variable $\boldsymbol{Y} \in R^n$, the way we estimate the parameter β is to minimize the following objective function:

$$\min_\beta \quad \sum_{i=1}^{n} l_\tau(Y_i - \boldsymbol{\beta}^T \boldsymbol{X}_{(\boldsymbol{i},)}).$$

$\boldsymbol{X}_{(\boldsymbol{i},)}$ denotes the i-th row of $\boldsymbol{X}$. And the function $l_\tau(\cdot)$ is defined by

$$l_\tau(t) = \begin{cases} \tau t, & t \geq 0, \\ (\tau - 1)t, & t < 0. \end{cases}$$

The quantile τ is determined based on the interest of the concrete problems. For example, we set $\tau = 0.99$ to estimate water level when we plan to build a dam that could prevent flood once in a hundred years. The minimizer of the above objective function would be our estimate of $\boldsymbol{\beta}$.

(a) (15 pts) Formulate the optimization problem into a linear programming.

Answer

Replacing the absolute value functions by inequalities, we could write it out as an LP:

$$\max_{u,v,\beta} \quad \tau \sum_{i=1}^{n} u_i + (1-\tau) \sum_{i=1}^{n} v_i \tag{1}$$

$$\text{subject to} \quad \boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{u} - \boldsymbol{v}, \tag{2}$$

$$\boldsymbol{u} \geq 0, \tag{3}$$

$$\boldsymbol{v} \geq 0. \tag{4}$$

(b) (10 pts) Show that there exists an optimal $\boldsymbol{\beta^*}$ with a line passing at least p data points. In other words, $l_\tau(Y_i - \boldsymbol{\beta^*}^T \boldsymbol{X_{(i,)}}) = 0$ for at least p different indices.

Answer

For linear programming problem, there exists at least one optimal solution being extreme point. The formula (2), (3), (4) altogether form a polyhedron. There are $2n$ inequality constraints, n equality constraints and totally $2n + p$ variables. For the extreme point, there will be at least $n + p$ inequalities in (3) and (4) that equality holds. It means that there will be at least p pairs of $(u_i, v_i) = (0, 0)$. So the optimal regression line will pass through the data samples with these indices.

Question 4 (20 pts)

In this question, we consider the problem of robust linear programming problem.

$$\min_{x} \sup_{c\in\mathcal{C}} c^T x,$$
$$\text{subject to } Ax \geq b,$$

with variable $x \in R^n$ and the set $\mathcal{C} = \{c \mid Fc \leq g, Ec = h\}$. We use the set $\mathcal{C}$ to charaterize the uncertainty for the parameters in the real world settings due to error of measurement and randomness. The objective function is to minimize the superior value over all the choice of c in $\mathcal{C}$, that is, to obtain a worst case optimality.

(a) (10 pts) Find the dual of the problem

$$\max_{c} c^T x$$
$$\text{subject to } Fc \leq g, Ec = h$$

with decision variable c. (You can regard x as a known vector here.)

Answer

Let p be the dual variables for the inequality constraints and q for the equality ones. Then its dual problem is:

$$\min_{p,q} g^T p + h^T q$$
$$\text{subject to} \quad F^T p + E^T q = x$$
$$p \geq 0, q \text{ is free.}$$

(b) (10 pts) Use the result above to obtain a single LP formulation equivalent to the original robust LP.

Answer

The problem would be:

$$\min_{p,q} g^T p + h^T q$$
$$\text{subject to} \quad Ax \geq b$$
$$F^T p + E^T q = x$$
$$p \geq 0, q \text{ is free.}$$

Bonus (10 pts)

During the the simplex method, if the current basic feasible solution is non-degenerate and there is only one negative reduced cost coefficient, then the entering variable will remain as a basic variable for the rest of the simplex method procedures.

Answer

Assume at the moment, the LP problem takes the form:

$$\min \quad c_0 + \sum_{i=1}^{k} c_i x_i,$$

$$\text{subject to } Ax = b, x \geq 0,$$

where $c_1 < 0$ and $c_2, ..., c_k \geq 0$. If x_1 is not a basic variable in the future, the objective value will be no smaller than c_0. However, if we pivot x_1 to be a basic variable at the current step, we will get an objective value small than c_0. And consider the fact that the objective value keeps decreasing during the simplex method, x_1 will remain a basic variable.