HOMEWORK 1
Due JANUARY 26, 2007

Optional Reading. Read Luenberger’s *Linear and Nonlinear Programming* Chapter 1 and Appendices A and B, and Cottle’s *Class Note* Handouts 2 and 3.

Solve the following problems:

1. (10pts.) Plot the region specified by the inequalities:
   \[ \|x\| \geq 1, \quad |x_i| \leq 1 \quad (i = 1, 2) \]
   Is the set of points satisfying these constraints non–empty? Is it closed? Is it bounded? Is it convex?

2. (10pts.) Consider the function \( f : \mathbb{R}_+ \to \mathbb{R} \) defined by:
   \[
   f(x) = \begin{cases} 
   0 & \text{if } x = 0 \\
   x \ln x & \text{if } x > 0
   \end{cases}
   \]
   Is this function continuous? convex? Does it have a minimizer on the positive real line? Justify your answer.

3. (15pts.) Let \( f : \mathbb{R}_+^n \to \mathbb{R} \) be a given convex function. Show that the function \( g : \mathbb{R}_+^{n+1} \to \mathbb{R} \) given by \( g(\tau; x) = \tau \cdot f(x/\tau) \) (called the homogenized version of \( f \)) is also a convex function in the domain of \( (\tau; x) \in \mathbb{R}_+^{n+1} \). Now, suppose that \( f \) is twice–differentiable. Write out the gradient vector and the Hessian matrix of \( g \).

4. (15pts.) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a convex function on \( \mathbb{R}^n \), and let \( g \) be a convex non–decreasing function on \( \mathbb{R} \), i.e. for all \( x, y \in \mathbb{R} \), \( x \leq y \) implies that \( g(x) \leq g(y) \).
   
   (a) Show that the composite function \( h = g \circ f \) is convex on \( \mathbb{R}^n \).
   
   (b) Show that the result established in (a) is not valid without the assumption that \( g \) is a non–decreasing function.

5. (25pts.) Show that \( f : \mathbb{R}^n \to \mathbb{R} \) is both convex and concave iff \( f(x) = c^T x + d \) for some constant vector \( c \in \mathbb{R}^n \) and scalar \( d \in \mathbb{R} \).
6. (10pts.) Consider the set \( S = \{ x \in \mathbb{R}^2 : |x_i| \leq 1 (i = 1, 2) \} \) and the functions:

\[
f_1(x) = 1 + x_1^2 + x_2^2 \quad \text{and} \quad f_2(x) = 1 - x_1^2 - x_2^2.
\]

Explain why these functions have maxima and minima on \( S \). What is the location and nature (i.e., local or global) of these maxima and minima?

7. (15pts.) Let \( f : (a, b) \to \mathbb{R} \) be a convex function. Show that for all \( s, t \) such that \( a < s < t < b \), we have:

\[
f\left(\frac{s + t}{2}\right) \leq \frac{1}{t - s} \int_s^t f(u) \, du \leq \frac{f(s) + f(t)}{2}
\]