Alternative Systems

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Linear Equations

Given matrix $A \in R^{m \times n}$ vector $b \in R^m$, exactly one of the following two systems is feasible:

$$Ax = b$$

or

$$A^T y = 0,\ b^T y = 1.$$

Polyhedron set

Given matrix $A \in R^{m \times n}$ vector $b \in R^m$, exactly one of the following two systems is feasible:

$$Ax = b, \ x \geq 0$$

or

$$A^T y \leq 0, \ b^T y = 1.$$

Proof: They cannot be both feasible since

$$0 \geq x^T(A^T y) = (x^T A^T)y = (Ax)^T y = b^T y = 1$$

which is a contradiction. Now we prove that if the "red" system is infeasible then the "green" system must be feasible.

Consider convex cone $C = \{Ax : \ x \geq 0\}$. We must have $b \notin C$. From the separating hyperplane theorem, there is $y \in R^m$ such that

$$y^T b > \sup_{x \geq 0} y^T Ax.$$

Thus, we must have i) $y^T b > 0$ since $x = 0$ makes $y^T Ax = 0$. ii) $y^T A \leq 0$ since, otherwise, say the first element of $y^T A$ is positive, we can choose $x = (\alpha; 0; ...; 0)$ and let $\alpha \to \infty$. Then, $y^T Ax \to \infty$ which is a contradiction to that $y^T Ax$ is bounded above for any $x \geq 0$.

More polyhedron set

Given matrix $A \in R^{m \times n}$ vector $c \in R^n$, exactly one of the following two systems is feasible:

$$A^T y \leq c$$

or

$$Ax = 0, \ c^T x = -1, \ x \geq 0.$$

Transformation: Rewrite the red system as

$$(A^T, \ -A^T, \ I) \begin{pmatrix} y' \\ y'' \\ s \end{pmatrix} = c, \ \begin{pmatrix} y' \\ y'' \\ s \end{pmatrix} \geq 0.$$

Then use the previous standard system to construct its alternative system.

Or rewrite the green system to

$$\begin{pmatrix} A \\ c^T \end{pmatrix} x = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \; x \geq 0.$$

Then use the previous standard system to construct its alternative system.

Strictly inequality

Given matrix $A \in R^{m \times n}$ vector $c \in R^n$, what's the alternative system for

$$c^T d < 0, \; Ad \geq 0?$$

For some positive constant ϵ, this system is equivalent to

$$c^T d \leq -\epsilon, \; -Ad \leq 0.$$

Thus, its alternative system is

$$\frac{1}{\epsilon} \cdot c - A^T x = 0, \; x \geq 0$$

or simply

$$c - A^T x = 0, \; x \geq 0.$$