Several Proofs on Conic LP

1.

Theorem 1. *Let E be a finite–dimensional Euclidean space equipped with the inner product $\bullet$, and let $a_1, \ldots, a_m \in E$. Let $C \subset E$ be a non–empty closed convex cone, and let $b \in \mathbb{R}^m$. Suppose that there exists an $\hat{y} \in \mathbb{R}^m$ such that $-\mathcal{A}^T\hat{y} \equiv -\sum_{i=1}^m \hat{y}_i a_i \in$ int(C^*). Then, the system:*

$$\mathcal{A}x \equiv (a_1 \bullet x, \ldots, a_m \bullet x) = b, \quad x \in C \tag{1}$$

has a solution $x \in C$ if and only if the system:

$$-\mathcal{A}^T y \in C^*, \quad b^T y = 1 \tag{2}$$

has no solution $y \in \mathbb{R}^m$.

Proof: We begin with some observations. First, we have $0 \in C$, since C is a non–empty closed cone. Next, recall that $C^* = \{z \in E : x \bullet z \geq 0 \text{ for all } x \in C\}$. We claim the following:

Lemma 1. *We have int$(C^*) = \{z \in E : x \bullet z > 0$ for all $x \in C\backslash\{0\}\}$.*

Proof. Suppose that $z \in \text{int}(C^*)$. Then, there exists an $\epsilon' > 0$ such that $z + \epsilon u \in C^*$ for all $u \in E$ with $u \bullet u = 1$ and all $\epsilon \in [0, \epsilon']$. In particular, we have $x \bullet z \geq 0$ and $x \bullet (z + \epsilon' u) \geq 0$ for all $x \in C$. Now, if $x \in C$ is such that $x \neq 0$ and $x \bullet z = 0$, then by taking $u = -x \in E$ we have $x \bullet (z + \epsilon' u) = -\epsilon'(x \bullet x) < 0$, which is a contradiction. Conversely, let $z \in E$ be such that $x \bullet z > 0$ for all $x \in C\backslash\{0\}$. Define $\epsilon' = \inf\{x \bullet z : x \in C,\ x \bullet x = 1\}$. Since the feasible region is compact, we see that the infimum is attained at some $x^* \in C\backslash\{0\}$, whence $\epsilon' > 0$. We now claim that $z + \epsilon u \in C^*$ for all $u \in E$ with $u \bullet u = 1$ and $\epsilon \in [0, \epsilon']$, which would then imply that $z \in \text{int}(C^*)$ as required. Indeed, using the bi–linearity of the inner product $\bullet$, for all $x \in C\backslash\{0\}$, we have:

$$\begin{aligned}
x \bullet (z + \epsilon u) &= \sqrt{x \bullet x}\left(\frac{x}{\sqrt{x \bullet x}} \bullet (z + \epsilon u)\right) \\
&\geq \sqrt{x \bullet x}\left(\epsilon' + \epsilon\frac{x}{\sqrt{x \bullet x}} \bullet u\right) && \text{(since } (x \bullet x)^{-1/2}(x \bullet z) \geq \epsilon' \text{ for all } x \in C\backslash\{0\}) \\
&\geq (\epsilon' - \epsilon) \cdot \sqrt{x \bullet x} && \text{(since } x \bullet u \geq -\sqrt{(x \bullet x)(u \bullet u)} \text{ and } u \bullet u = 1) \\
&\geq 0
\end{aligned}$$

This completes the proof of the claim and hence of the lemma. □

Now, we show that there does not exist $(x, y) \in C \times \mathbb{R}^m$ such that x solves (1) and y solves (2) simultaneously. Indeed, if $(x, y) \in C \times \mathbb{R}^m$ is such a pair, then by definition of C^*, we have:

$$0 \leq \left(-\mathcal{A}^T y\right) \bullet x = -\sum_{i=1}^{m} y_i (a_i \bullet x) = -\sum_{i=1}^{m} y_i b_i = -1$$

which is a contradiction. Now, suppose that system (1) has no solution. Define $K = \{\mathcal{A}x \in \mathbb{R}^m : x \in C\}$. Note that our hypothesis implies that $b \notin K$. We first show the following:

Lemma 2. *K is a non–empty closed convex set.*

Proof. It is clear that $0 \in K$, and the convexity of K follows from the convexity of C. Now, suppose that we have a sequence $b^i = \mathcal{A}x^i \in K$ such that $b^i \to \bar{b}$. We need to show that $\bar{b} \in K$. Note that the sequence $\{b^i\}$ is bounded, which in turn implies that $\{\hat{y}^T b^i\}$ is bounded for some $\hat{y} \in \mathbb{R}^m$ such that $-\mathcal{A}^T \hat{y} \in \text{int}(C^*)$. We claim that the sequence $\{x^i\}$ is bounded. Indeed, observe that:

$$-\hat{y}^T b^i = -\hat{y}^T \mathcal{A}x^i = -\sum_{j=1}^{m} \hat{y}_j (a_j \bullet x^i) = \left(-\sum_{j=1}^{m} \hat{y}_j a_j\right) \bullet x^i = -\mathcal{A}^T \hat{y} \bullet x^i$$

Now, if $x^i \neq 0$, then by the definition of $\hat{y}$ and Lemma 1, we have:

$$-\hat{y}^T b^i = -\mathcal{A}^T \hat{y} \bullet x^i = \left(\sqrt{x^i \bullet x^i}\right) \cdot \left(-\mathcal{A}^T \hat{y} \bullet \frac{x^i}{\sqrt{x^i \bullet x^i}}\right) \geq \delta \cdot \sqrt{x^i \bullet x^i}$$

for some $\delta > 0$. Since the leftmost quantity is bounded and is independent of x^i, it follows that the sequence $\{x^i\}$ is bounded as claimed. In particular, by the Bolzano–Weierstrass theorem, the sequence $\{x^i\}$ has a convergent subsequence whose limit we shall denote by $\bar{x}$. Note that $\bar{x} \in C$, since C is closed. It follows that $\bar{b} = \mathcal{A}\bar{x} \in K$, as desired. □

In order to complete the proof of Theorem 1, it remains to apply the Separating Hyperplane Theorem. Using Lemma 2 and the fact that $b \notin K$, we conclude the existence of an $s \in \mathbb{R}^m$ such that $b^T s > \sup\{z^T s : z \in K\}$. Since $0 \in K$, we see that $b^T s = \alpha > 0$. Now, for any $x \in C$, we have:

$$\left(-\mathcal{A}^T s\right) \bullet x = -\sum_{i=1}^{m} s_i (a_i \bullet x) = -s^T \mathcal{A}x$$

We claim that $s^T \mathcal{A}x \le 0$ for all $x \in C$. Suppose that this is not the case. Then, there exists an $x \in C$ such that $0 < s^T \mathcal{A}x \le \sup\{z^T s : z \in K\} < b^T s$, where the second inequality follows from the fact that $\mathcal{A}x \in K$. However, since C is a cone, we have $\gamma x \in C$ for all $\gamma > 0$. This implies that $0 < \gamma s^T \mathcal{A}x < b^T s$ for all $\gamma > 0$, which is impossible. Hence, we have $s^T \mathcal{A}x \le 0$ for all $x \in C$, whence $-\mathcal{A}^T s \in C^*$. Now, set $y = s/\alpha$. Then, we have $b^T y = 1$. Moreover, since C^* is a cone and $\alpha > 0$, we have $-\mathcal{A}^T y \in C^*$. This completes the proof.

2.

Theorem 2. *Let E be a finite–dimensional Euclidean space equipped with the inner product $\bullet$, and let $\mathbf{a}_1, \ldots, \mathbf{a}_m \in E$. Let $C \subset E$ be a non–empty closed convex cone, and let $\mathbf{b} \in \mathbb{R}^m$. Consider the Conic LP*

$$(CLP) \quad \begin{array}{ll} \textit{minimize} & \mathbf{c} \bullet \mathbf{x} \\ \textit{subject to} & \mathcal{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \in C. \end{array}$$

and its dual

$$(CLD) \quad \begin{array}{ll} \textit{maximize} & \mathbf{b} \bullet \mathbf{y} \\ \textit{subject to} & \mathcal{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{s} \in C^*. \end{array}$$

Let primal and dual feasible regions both be non-empty and have interior, that is, there is primal feasible $\mathbf{x}$ where $\mathbf{x} \in int(C)$ and dual feasible $(\mathbf{y}, \mathbf{s})$ where $\mathbf{s} \in int(C^*)$. Then, both primal and dual have optimal solutions with zero-duality gap, that is, there are x^* optimal for (CLP) and (y^*, s^*) optimal for (CLD) where

$$\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b} \bullet \mathbf{y}^*.$$

Proof: Given the conditions, we need prove that the system

$$(S) \quad \begin{array}{ll} \mathbf{b} \bullet \mathbf{y} - \mathbf{c} \bullet \mathbf{x} \ge 0 & \\ \mathcal{A}\mathbf{x} = \mathbf{b} & \mathbf{x} \in C \\ -\mathcal{A}^T\mathbf{y} - \mathbf{s} = -\mathbf{c}, & \mathbf{s} \in C^* \end{array}$$

always has a solution.

From Theorem 1, the alternative system to this feasibility problem, since we are given there is primal feasible $\mathbf{x}$ where $\mathbf{x} \in \mathrm{int}(C)$ and dual feasible $(\mathbf{y}, \mathbf{s})$ where $\mathbf{s} \in \mathrm{int}(C^*)$, is

$$(AS) \quad \begin{array}{ll} \mathbf{b} \bullet \mathbf{y}' - \mathbf{c} \bullet \mathbf{x}' = 1 & \\ \mathcal{A}\mathbf{x}' - \tau\mathbf{b} = \mathbf{0}, & \mathbf{x}' \in C \\ -\mathcal{A}^T\mathbf{y}' - \mathbf{s}' + \tau\mathbf{c} = \mathbf{0}, & \mathbf{s}' \in C^* \\ \tau > 0. & \end{array}$$

We now prove that the alternative system (AS) has no solution by contradiction. Let $(\tau, \mathbf{x}'\mathbf{y}'\mathbf{s}')$ be a solution to (AS). Consider two cases

Case 1: $\tau = 0$. In this case, we have

$$(AS) \quad \begin{array}{ll} \mathbf{b} \bullet \mathbf{y}' - \mathbf{c} \bullet \mathbf{x}' = 1 & \\ \mathcal{A}\mathbf{x}' = \mathbf{0}, & \mathbf{x}' \in C \\ -\mathcal{A}^T\mathbf{y}' - \mathbf{s}' = \mathbf{0}, & \mathbf{s}' \in C^*. \end{array}$$

Thus, either $\mathbf{c} \bullet \mathbf{x}' < 0$ or $\mathbf{b} \bullet \mathbf{y}' > 0$ or both. With out loss of generality, assume that $\mathbf{c} \bullet \mathbf{x}' < 0$ and let $\bar{\mathbf{x}}$ be any feasible solution for (CLP). Then, for any $\alpha \geq 0$, $\mathbf{x} + \alpha\mathbf{x}'$ is also a feasible solution and its objective value is

$$\mathbf{c} \bullet (\mathbf{x} + \alpha\mathbf{x}') = \mathbf{c} \bullet \mathbf{x} + \alpha\mathbf{c} \bullet \mathbf{x}'.$$

Let α goes to ∞, $\mathbf{c} \bullet (\mathbf{x} + \alpha\mathbf{x}')$ will be unbounded from below, which contradicts the Weak Duality Theorem, since the dual (CLD) is feasible.

Case 1: $\tau > 0$. In this case, $\mathbf{x}'/\tau$ and $(\mathbf{y}', \mathbf{s}')/\tau$ are feasible solution for (CLP) and (CLD), respectively. Thus, from the Weak Duality Theorem

$$\mathbf{c} \bullet \mathbf{x}'/\tau - \mathbf{b} \bullet \mathbf{y}'/\tau \geq 0$$

or

$$\mathbf{c} \bullet \mathbf{x}' - \mathbf{b} \bullet \mathbf{y}' \geq 0$$

which contradicts to the first equality of (AS).

Thus, (S) must have a solution, which is the desired theorem.