Readings. Read Luenberger’s *Linear and Nonlinear Programming* 2.5 and 4.1-4.2 and Appendix B, the supplemental text 12.10, and Cottle’s *Class Note*, and my Handouts 2, 3, 4 and 5.

Solve the following problems.

1. (20pts.) Worst-case probability estimation (the Markov inequality).

   (a) Consider a probability distribution defined on a population of finite points \( \{\xi^k \in \mathbb{R} \} \), \( k = 1, \ldots, n \). However, you don’t know the distribution but the mean \( \mu \) and a upper bound \( v (> 0) \) on the variance of the random numbers. Given an interval \( I \), you want to estimate the maximum probability of the random number being in \( I \). Formulate the problem as a linear program and construct its dual. This estimation is referred as the Markov inequality.

   (b) Now the points are random vectors \( \{\xi^k \in \mathbb{R}^m \} \), and you only know the mean vector \( \mu \in \mathbb{R}^m \) and a upper bound (in the positive semidefinite sense) \( V(\succeq 0) \) on the covariance matrix of the random vectors. Given a set \( I \) in \( \mathbb{R}^m \), you want to estimate the maximum probability of the random vector being in \( I \). Formulate the problem as a conic linear program and construct its dual.

2. (20pts.)

   (a) Recall that the dual cone \( K^* \) of a convex cone \( K \) is defined as \( K^* = \{ y \in \mathbb{R}^n : x^T y \geq 0 \text{ for all } x \in K \} \). Show that the dual cone of

   \[
   K_o = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq x_2 \leq \cdots \leq x_n\}
   \]

   where \( n \geq 2 \), is given by:

   \[
   K^*_o = \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : \sum_{j=1}^{k} y_j \leq 0 \text{ for } k = 1, \ldots, n-1; \sum_{j=1}^{n} y_j = 0 \right\}
   \]
(b) Consider problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x_1 = 1, \\
& \quad x_n = n, \\
& \quad x \in K_o
\end{align*}
\]

What is the dual of the problem?

3. (20pts.) Consider convex cone

\[ C = \{(t; x) : t > 0, \ t c(x/t) \leq 0, \ x \in K \subset R^2\}, \]

where \( c(x) \) is a convex function over a convex cone \( K \). Construct the dual cone of \( C \) for each of the following \( c(x) \):

(a) \( c(x) = \|x\|_p - 1 \) and \( K = R^2, \ p = 1, 2, ... \).

(b) \( c(x) = x^T Q x - 1 \) and \( K = R^2 \), where \( Q \) is a (symmetric) positive definite matrix.

(c) \( c(x) = \sum_{i=1}^{2} e^{x_i} - 2 \) and \( K = R^2 \).

(d) \( c(x) = \sum_{i=1}^{2} \frac{1}{x_i} - 1 \) and \( K = R^2_{++} \).

4. (20pts.)

(a) Consider the minimal–objective function \( c \mapsto z(c) \) for fixed \( A \) and \( b \), assuming the CLP is strictly feasible:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad A x = b \\
& \quad x \in C
\end{align*}
\]

for a closed convex cone. Show that \( z(\cdot) \) is a concave function in the vector \( c \) where \( z(c) \) is finite and attained at an optimal solution.

(b) Let \( c \) be any given \( n \)-dimensional vector and \( \lambda_k(c) \) be the \( k \)-th smallest element of \( c \), where \( k < n \). Then, prove the following statement

\[
\begin{align*}
\lambda_1(c) + \cdots + \lambda_k(c) = \minimize & \quad c^T x \\
\text{subject to} & \quad e^T x = k; \quad 0 \leq x \leq e.
\end{align*}
\]

Thus, the \textit{sum of \( k \) smallest elements function} of \( c \) is a concave function in \( c \).

5. (20pts.) Suppose your next-year before-tax income is \( M \); and your total health-care cost, \textit{out of your own pocket}, for the next year is a random number \( h \) with a bounded probability density function \( f(h) \). The problem is how much pre-tax money you should decide now to put into your health-care flexible spending account to cover your next
year-long health care cost. The good thing about the spending account is that it is the pre-tax money so as tax-free, the bad thing is that any remaining amount will be gone at the end of year. Suppose you put $x$ amount into the spending account and your tax rate is pretty much fixed at $\alpha \times 100\%$.

(a) Show that your expected after-tax net-income, excluding health-care cost, is a concave function in $x$.

(b) Find what $x$ should be such that it maximizes the expected after-tax net-income.

(c) Suppose you don’t know the distribution function $f(h)$ but only $h$ in the range $[a, b]$. Then how do you decide $x$? You may use any strategy: max-min, min-regret, etc.

6. (20pts.) Luenberger’s *Linear and Nonlinear Programming*: page 110-111, Problem 7 on economic free competition.

7. (20pts.) Luenberger’s *Linear and Nonlinear Programming*: page 111-112, Problem 8 on bi-matrix game theory.