ASSIGNMENT 3
Due February 29, 2008

Readings. Read Luenberger’s Linear and Nonlinear Programming Chapters 6.1-6.5 and 10.1-10.9, and Cottle’s Class Note Handouts 6, 7 and 8; and review my Lecture Series 5, 6, 7 and 8.

Solve the following problems.

1. Consider a quadratic programming problem of the form
   \[
   \begin{align*}
   \text{minimize} \quad & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 - 2x_2 \\
   \text{subject to} \quad & x_1 + x_2 - \kappa \geq 0 \\
   \quad & x_1, x_2 \geq 0
   \end{align*}
   \]
   where \(\kappa \in \mathbb{R}\) is a constant.
   (a) Give a geometric interpretation of an instance of this problem.
   (b) Why does such a problem always have an optimal solution?
   (c) Using the KKT conditions, verify that (1.5, 2.5) solves the instance of this quadratic programming problem in which \(\kappa = 4\).
   (d) For certain values of \(\kappa\) the optimal solution of the problem lies on the boundary of the feasible region. What are those values, and what are the corresponding optimal solutions, Lagrange multipliers, and optimal objective function values?
   (e) For the conditions described in (d), compare the value of the Lagrange multiplier corresponding to the constraint \(x_1 + x_2 - \kappa \geq 0\) and the derivative of the objective function with respect to \(\kappa\).
   (f) What is the optimal solution for an arbitrary instance of this problem (i.e., for arbitrary \(\kappa\)) for which the optimal solution does not lie on the boundary of the feasible region?

2. Consider a nonlinear program
   \[
   \begin{align*}
   \text{minimize} \quad & f(x) \\
   \text{subject to} \quad & x \geq 0.
   \end{align*}
   \]
Assume that $f$ is differentiable and convex on $\mathbb{R}^n$. Verify that the vector $\bar{x}$ is globally optimal if and only if
\[ \nabla f(\bar{x}) \geq 0 \]
\[ \bar{x} \geq 0 \]
\[ \bar{x}^T \nabla f(\bar{x}) = 0 \]
(Systems of this form are called (nonlinear) complementarity problems.)

3. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be separable if it can be written in the form
\[ f(x_1, \ldots, x_n) = \sum_{j=1}^{n} f_j(x_j). \]
Such functions were studied by J.W. Gibbs in connection with work on the chemical equilibrium problem (1876). He showed that (in the differentiable case) a necessary condition of local optimality for a feasible point $\bar{x}$ of the separable nonlinear programming problem
\[
\text{minimize} \quad \sum_{i=1}^{n} f_i(x_i) \\
\text{subject to} \quad \sum_{i=1}^{n} x_i = M \quad (M > 0) \\
\quad x_i \geq 0 \quad i = 1, \ldots, n
\]
is that there exist a number $\bar{\lambda}$ such that
\[
\frac{\partial f_i(\bar{x}_i)}{\partial x_i} = \bar{\lambda} \quad \text{if} \quad \bar{x}_i > 0 \\
\frac{\partial f_i(\bar{x}_i)}{\partial x_i} \geq \bar{\lambda} \quad \text{if} \quad \bar{x}_i = 0
\]
Use the first-order optimality conditions to show this theorem.


5. Consider a generalized Arrow-Debreu equilibrium problem where the market has $n$ agents and $m$ type commodities. Agent $i$, $i = 1, \ldots, n$, has a boundle amount of $w_i = (w_{i1}; w_{i2}; \ldots; w_{im}) \in \mathbb{R}^m$ commodities initially and has a linear utility function with coefficients $u_i = (u_{i1}; u_{i2}; \ldots; u_{im}) > 0 \in \mathbb{R}^m$. The problem is how to price each commodity so that the market clears. Note that, given the price vector $p = (p_1; p_2; \ldots; p_m) > 0$, the individual utility maximization problem is
\[
\max \quad u_i^T x_i \\
\text{s.t.} \quad p^T x_i \leq p^T w_i \\
x_i \geq 0.
\]
Show the followings.
(a) For given $p$, write down the optimality condition of the individual utility maximization problem.

(b) If $p$ and $x_i, \ i = 1, ..., n$, satisfy the constraints

\[
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i,
\]

\[
\frac{u^T_i x_i}{u_{ij}} \geq \frac{p^T w_i}{p_j}, \ \forall i, j
\]

\[
x_{ij} \geq 0, \ p_j > 0, \ \forall i, j,
\]

Then, $p$ is an equilibrium price vector.

(c) To find a feasible point, a minimization problem is formed as follows:

\[
\min \ \theta
\]

s.t. \[
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i + e^\theta,
\]

\[
\frac{u^T_i x_i}{u_{ij}} \geq \frac{p^T w_i}{p_j}, \ \forall i, j
\]

\[
x_{ij} \geq 0, \ p_j > 0, \ \forall i, j.
\]

Verify that the problem is feasible and the minimal value of the problem is 0.

(d) By introducing new variable $y_j = \log(p_j), \ j = 1, ..., m$, the problem can be written

\[
\min \ \theta
\]

s.t. \[
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i + e^\theta,
\]

\[
\log(u^T_i x_i) - \log(u_{ij}) \geq \log \left( \sum_{k=1}^{m} w_{ik} e^{y_k} \right) - y_j, \ \forall i, j
\]

\[
x_{ij} \geq 0, \ p_j > 0, \ \forall i, j.
\]

Show this problem is convex in $x_{ij}$ and $y_j$. (Hint: $\log(\sum_{k=1}^{m} w_{ik} e^{y_k})$ is a convex function in $y_k$’s.)