Elements of Mathematical Analysis and Conic Duality

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Appendix B, Chapters 4.1-2, 6.1-6.4
The following theorem states that a polyhedral cone can be generated by a set of basic directional vectors.

**Theorem 1** Given matrix $A \in \mathbb{R}^{m \times n}$, let convex polyhedral cone $C = \{Ax : x \geq 0\}$. For any $b \in C$,

$$b = \sum_{i=1}^{d} a_{j_i} x_{j_i}, \quad x_{j_i} \geq 0, \forall i$$

for some linearly independent vectors $a_{j_1}, ..., a_{j_d}$ chosen from $a_1, ..., a_n$.

There is a construct proof of the theorem (page 21 of the text).
Basic and Basic Feasible Solution I

Now consider the feasible set \( \{ x : A x = b, \ x \geq 0 \} \) for given data \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Select \( m \) linearly independent columns, denoted by the variable index set \( B \), from \( A \). Solve \( A_B x_B = b \) for the \( m \)-dimension vector \( x_B \), and set the remaining variables, \( x_N \), to zero. Then, we obtain a solution \( x \) such that \( A x = b \), that is called a basic solution to with respect to the basis \( A_B \). If a basic solution \( x_B \geq 0 \), then \( x \) is called a basic feasible solution, or BFS.

Carathéodory’s theorem implies that

**Theorem 2** If there is a feasible solution \( x \) to \( \{ x : A x = b, \ x \geq 0 \} \), then there is a basic feasible solution to the system (page 21 of the text), and it is an extreme or corner point of the feasible set and vice versa.

**Corollary 1** The set \( \{ x : A x = b, \ x \geq 0 \} \) is a polyhedral set.
The most important type of convex set is hyperplane, also called linear variety or affine set: if for any two points are in \( H \) then their affine combination is also in \( H \).

Hyperplanes dominate the entire theory of optimization. Let \( \mathbf{a} \) be a nonzero \( n \)-dimensional (slope) vector, and let \( b \) be a real (intercept) number. The set

\[
H = \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} = b \}
\]

is a hyperplane in \( \mathbb{R}^n \). Relating to hyperplane, upper and lower closed half spaces are given by

\[
H_+ = \{ \mathbf{x} : \mathbf{a} \cdot \mathbf{x} \geq b \}
\]

\[
H_- = \{ \mathbf{x} : \mathbf{a} \cdot \mathbf{x} \leq b \}.
\]
The most important theorem about the convex set is the following separating hyperplane theorem (page 510 of the text).

**Theorem 3** (Separating hyperplane theorem) Let $C$ be a closed convex set in $\mathbb{R}^m$ and let $b$ be a point exterior to $C$. Then there is a vector $y \in \mathbb{R}^m$ such that

$$b \cdot y > \sup_{x \in C} x \cdot y.$$ 

**Theorem 4** (Supporting hyperplane theorem) Let $C$ be a closed convex set and let $b$ be a point on the boundary of $C$. Then there is a vector $y \in \mathbb{R}^m$ such that

$$b \cdot y = \sup_{x \in C} x \cdot y.$$ 

Let $C$ be a unit circle centered at point $(1; 1)$. That is, $C = \{x \in \mathbb{R}^2 : (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1\}$. If $b = (2; 0)$, $y = (1; -1)$ is a separating hyperplane vector. If $b = (0; -1)$, $y = (0; -1)$ is a separating hyperplane vector. It is worth noting that these separating hyperplanes are not unique.
Figure 1: Illustration of the separating hyperplane theorem; an exterior point $b$ is separated by a hyperplane from a convex set $C$. 
The following results are **Farkas’ lemma** and its variants.

**Theorem 5** Let $A \in \mathcal{R}^{m \times n}$ and $b \in \mathcal{R}^m$. Then, the system $\{x : Ax = b, \ x \geq 0\}$ has a feasible solution $x$ if and only if its alternative system $-A^T y \geq 0$ and $b^T y > 0$ has no feasible solution $y$.

Geometrically, Farkas’ lemma means that if a vector $b \in \mathcal{R}^m$ does not belong to the convex cone generated by $a_1, \ldots, a_n$, then there is a hyperplane separating $b$ from $\text{cone}(a_1, \ldots, a_n)$.

**Example** Let $A = (1, 1)$ and $b = -1$. Then, there is $y = -1$ such that $-A^T y \geq 0$ and $by > 0$.
Proof

Let \( \{x : Ax = b, \ x \geq 0\} \) have a feasible solution, say \( \bar{x} \). Then, \( \{y : A^T y \leq 0, \ b^T y > 0\} \) is infeasible, since otherwise,

\[
0 < b^T y = (Ax)^T y = x^T (A^T y) \leq 0
\]

from \( x \geq 0 \) and \( A^T y \leq 0 \).

Now let \( \{x : Ax = b, \ x \geq 0\} \) have no feasible solution, or \( b \not\in C := \{Ax : x \geq 0\} \). We now prove that its alternative system has a solution. We first prove

Lemma 1 \( C = \{Ax : x \geq 0\} \) is a closed convex set.

That is, any convergent sequence \( b^k \in C, \ k = 1, 2, \ldots\) has its limit point \( \bar{b} \) also in \( C \). Let \( b^k = Ax^k, \ x^k \geq 0 \). Then by Carathéodory’s theorem, we must have \( b^k = A_{B^k} x_{B^k}, \ x_{B^k} \geq 0 \) where \( A_{B^k} \) is a basis of \( A \). Therefore, \( x_{B^k} \), together with zero values for the nonbasic variables, is bounded for all \( k \), so that it has sub-sequence, say indexed by \( l = 1, \ldots \), where \( x^l = x_{B^l} \) has a limit point \( \bar{x} \) and \( \bar{x} \geq 0 \). Consider this very sub-sequence \( b^l = Ax^l \) we must also have \( b^l \to \bar{b} \). Then from

\[
||\bar{b} - A\bar{x}|| = ||\bar{b} - b^l + Ax^l - A\bar{x}|| \leq ||\bar{b} - b^l|| + ||Ax^l - A\bar{x}|| \leq ||\bar{b} - b^l|| + ||A|| ||x^l - \bar{x}||
\]
we must have $\mathbf{b} = A\mathbf{x}$, that is, $\mathbf{b} \in C$; since otherwise the right-hand side of the above inequality is strictly greater than zero which is a contradiction.

Now since $C$ is a closed convex set, by the separating hyperplane theorem, there is $\mathbf{y}$ such that

$$\mathbf{y} \cdot \mathbf{b} > \sup_{\mathbf{c} \in C} \mathbf{y} \cdot \mathbf{c}$$

or

$$\mathbf{y} \cdot \mathbf{b} > \sup_{x \geq 0} \mathbf{y} \cdot (A\mathbf{x}) = \sup_{x \geq 0} A^T \mathbf{y} \cdot \mathbf{x}. \quad (1)$$

From $0 \in C$ we have $\mathbf{y} \cdot \mathbf{b} > 0$.

Furthermore, $A^T \mathbf{y} \leq 0$. Since otherwise, say $(A^T \mathbf{y})_1 > 0$, one can have a vector $\mathbf{x} \geq 0$ such that $\bar{x}_1 = \alpha > 0, \bar{x}_2 = \ldots = \bar{x}_n = 0$, from which

$$\sup_{x \geq 0} A^T \mathbf{y} \cdot \mathbf{x} \geq A^T \mathbf{y} \cdot \mathbf{x} = (A^T \mathbf{y})_1 \cdot \alpha$$

and it tends to $\infty$ as $\alpha \to \infty$. This is a contradiction because $\sup_{x \geq 0} A^T \mathbf{y} \cdot \mathbf{x}$ is bounded from above by $(1)$.
**Farkas’ Lemma Variant**

**Theorem 6** Let $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^n$. Then, the system $\{y : c - A^T y \geq 0\}$ has a solution $y$ if and only if that $Ax = 0$, $x \geq 0$, and $c^T x < 0$ has no feasible solution $x$.

**Example** Let $A = (1; -1)$ and $c = (1; -2)$. Then, there is $x = (1; 1) \geq 0$ such that $Ax = 0$ and $c^T x < 0$. 
A vector $y$, with $A^T y \leq 0$ and $b^T y = 1(>0)$, is called an infeasibility certificate for the system 
\{ $x : Ax = b, \ x \geq 0$ \}. 

**Alternative System Pair I** 

\[ Ax = b, \ x \geq 0. \] 

\[ -A^T y \geq 0, \ b^T y = 1(>0) \]
A vector $x$, with $Ax = 0$, $x \geq 0$ and $c^T x = -1$, is called an infeasibility certificate for the system \{y : c - A^T y \geq 0\}. 

**Alternative System Pair II**

$$Ax = 0, \ x \geq 0, \ c^T x = -1(<0).$$

$$c - A^T y \geq 0$$
Farkas’ Lemma for General Closed Convex Cones?

\{x : Ax = b, \ x \in K\}

and

\{y : -A^T y \in K^*, \ b^T y > 0\}.

Or in operator form: given data vector or matrix \(a_i, i = 1, \ldots, m\), and \(b \in \mathbb{R}^m\), an “alternative” system pair would be

\[Ax = b, \ x \in K,\]

and

\[-A^T y \in K^*, \ b^T y = 1(> 0)\]

where

\[Ax = (a_1 \cdot x; \ldots; a_m \cdot x) \in \mathbb{R}^m \text{ and } A^T y = \sum_{i} y_i a_i.\]

They are for general convex cone \(K\)?
An SDP Cone Example when “Alternative System” Failed

\[ K = S^2_+. \]

\[ \mathbf{a}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

and

\[ \mathbf{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \]

The Problem: \( C := \{ Ax : x \in K \} \) is not closed even when \( K \) is a closed convex cone.
Let $K$ be a closed and convex cone in the rest of the course.

If there is $y$ such that $-A^T y \in \text{int} \ K^*$, then $C := \{ Ax : x \in K \}$ is a closed convex cone. Consequently,

$$Ax = b, \quad x \in K,$$

and

$$-A^T y \in K^*, \quad b^T y = 1 (> 0)$$

are an alternative system pair.

And if there is $x$ such that $A^T x = 0, \ x \in \text{int} \ K$, then

$$Ax = 0, \quad x \in K, \quad c \cdot x = -1 (< 0)$$

and

$$c - A^T y \in K^*$$

are an alternative system pair.
Recall Conic LP

\[(CLP) \quad \text{minimize} \quad c \cdot x\]

subject to \( a_i \cdot x = b_i, \ i = 1, 2, \ldots, m, \ x \in K, \)

where \( K \) is a closed and pointed convex cone.

Linear Programming (LP): \( c, a_i, x \in \mathcal{R}^n \) and \( K = \mathcal{R}_+^n \)

Second-Order Cone Programming (SOCP): \( c, a_i, x \in \mathcal{R}^n \) and \( K = SOC = \{ x : x_1 \geq \|x_{-1}\|_2 \} \).

Semidefinite Programming (SDP): \( c, a_i, x \in \mathcal{S}^n \) and \( K = S_+^n \)

p-Order Cone Programming (POCP): \( c, a_i, x \in \mathcal{R}^n \) and \( K = POC = \{ x : x_1 \geq \|x_{-1}\|_p \} \).

Here, \( x_{-1} \) is the vector \((x_2; \ldots; x_n) \in \mathcal{R}^{n-1}\).

Cone \( K \) can be also a product of different cones, that is, \( x = (x_1; x_2; \ldots) \) where \( x_1 \in K_1, x_2 \in K_2, \ldots \) and so on.
LP, SOCP, and SDP Examples Again

\((LP)\) minimize \(2x_1 + x_2 + x_3\)
subject to \(x_1 + x_2 + x_3 = 1,\)
\((x_1; x_2; x_3) \geq 0.\)

\((SOCP)\) minimize \(2x_1 + x_2 + x_3\)
subject to \(x_1 + x_2 + x_3 = 1,\)
\(x_1 - \sqrt{x_2^2 + x_3^2} \geq 0.\)

\((SDP)\) minimize \(2x_1 + x_2 + x_3\)
subject to \(x_1 + x_2 + x_3 = 1,\)
\[
\begin{pmatrix}
x_1 & x_2 \\
x_2 & x_3
\end{pmatrix} \succeq 0.
\]
(SDP) can be rewritten as

minimize

\[
\begin{pmatrix}
  2 & .5 \\
  .5 & 1 \\
  1 & .5 \\
  .5 & 1 \\
\end{pmatrix}
\cdot
\begin{pmatrix}
  x_1 & x_2 \\
  x_2 & x_3 \\
  x_1 & x_2 \\
  x_2 & x_3 \\
\end{pmatrix}
= 1,
\]

subject to

\[
\begin{pmatrix}
  x_1 & x_2 \\
  x_2 & x_3 \\
\end{pmatrix} \succeq 0,
\]

that is

\[
c = \begin{pmatrix}
  2 & .5 \\
  .5 & 1 \\
\end{pmatrix}
\quad \text{and} \quad
a_1 = \begin{pmatrix}
  1 & .5 \\
  .5 & 1 \\
\end{pmatrix}.
\]
The dual problem to

\[(CLP) \quad \text{minimize} \quad c \cdot x\]

subject to \[a_i \cdot x = b_i, \; i = 1, 2, \ldots, m, \; x \in K.\]

is

\[(CLD) \quad \text{maximize} \quad b^T y\]

subject to \[\sum_i^m y_i a_i + s = c, \; s \in K^*,\]

where \(y \in \mathcal{R}^m\), \(s\) is called the dual slack vector/matrix, and \(K^*\) is the dual cone of \(K\). The former is called the primal problem, and the latter is called dual problem.

**Theorem 7** The dual of the dual is the primal.
LP, SOCP, and SDP Examples

\[
\begin{align*}
\min & \quad (2; 1; 1)^T x \\
\text{s.t.} & \quad e^T x = 1, \quad x \geq 0.
\end{align*}
\]

\[
\begin{align*}
\max & \quad y \\
\text{s.t.} & \quad e \cdot y + s = (2; 1; 1), \quad s \geq 0.
\end{align*}
\]

\[
\begin{align*}
\min & \quad (2; 1; 1)^T x \\
\text{s.t.} & \quad e^T x = 1, \quad x_1 - \|x_{-1}\| \geq 0.
\end{align*}
\]

\[
\begin{align*}
\max & \quad y \\
\text{s.t.} & \quad e \cdot y + s = (2; 1; 1), \quad s_1 - \|s_{-1}\| \geq 0.
\end{align*}
\]
\[
\begin{align*}
\min & \quad \begin{pmatrix} 2 & .5 \\ .5 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = 1, \\
\text{s.t.} & \quad x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0,
\end{align*}
\]

\[
\begin{align*}
\max & \quad y \\
\text{s.t.} & \quad \begin{pmatrix} 1 & .5 \\ .5 & 1 \end{pmatrix} y + s = \begin{pmatrix} 2 & .5 \\ .5 & 1 \end{pmatrix}, \\
& \quad s = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \succeq 0.
\end{align*}
\]
Recall the cost-to-go value of the reinforcement learning LP problem:

\[
\begin{align*}
\text{maximize}_y & \quad \sum_{i=1}^{m} y_i \\
\text{subject to} & \quad y_1 - \gamma \mathbf{p}_j^T y \leq c_j, \ j \in \mathcal{A}_1 \\
& \quad \vdots \\
& \quad y_i - \gamma \mathbf{p}_j^T y \leq c_j, \ j \in \mathcal{A}_i \\
& \quad \vdots \\
& \quad y_m - \gamma \mathbf{p}_j^T y \leq c_j, \ j \in \mathcal{A}_m.
\end{align*}
\]

\[
\begin{align*}
\text{minimize}_x & \quad \sum_{j \in \mathcal{A}_1} c_j x_j + \cdots + \sum_{j \in \mathcal{A}_m} c_j x_j \\
\text{subject to} & \quad \sum_{j \in \mathcal{A}_1} (\mathbf{e}_1 - \gamma \mathbf{p}_j) x_j + \cdots + \sum_{j \in \mathcal{A}_m} (\mathbf{e}_m - \gamma \mathbf{p}_j) x_j = \mathbf{e}, \\
& \quad \vdots \\
& \quad x_j \quad \vdots \\
& \quad \geq 0, \ \forall j,
\end{align*}
\]

where \( \mathbf{e}_i \) is the unit vector with 1 at the \( i \)th position and 0 everywhere else.
Variable $x_j, j \in A_i$, is the state-action frequency or called flux, or the expected present value of the number of times that an individual is in state $i$ and takes state-action $j$.

Thus, solving the problem entails choosing a state-action frequencies/fluxes that minimizes the expected present value of total costs for the infinite horizon, where the RHS is $(1; 1; 1; 1; 1; 1)$:

<table>
<thead>
<tr>
<th>x:</th>
<th>(0₁)</th>
<th>(0₂)</th>
<th>(1₁)</th>
<th>(1₂)</th>
<th>(2₁)</th>
<th>(2₂)</th>
<th>(3₁)</th>
<th>(3₂)</th>
<th>(4₁)</th>
<th>(5₁)</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>c:</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>(0)</td>
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<td>1</td>
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<tr>
<td>(1)</td>
<td>$-\gamma$</td>
<td>0</td>
<td>1</td>
<td>1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(2)</td>
<td>0</td>
<td>$-\gamma/2$</td>
<td>$-\gamma$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(3)</td>
<td>0</td>
<td>$-\gamma/4$</td>
<td>0</td>
<td>$-\gamma/2$</td>
<td>$-\gamma$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(4)</td>
<td>0</td>
<td>$-\gamma/8$</td>
<td>0</td>
<td>$-\gamma/4$</td>
<td>0</td>
<td>$-\gamma/2$</td>
<td>$-\gamma$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(5)</td>
<td>0</td>
<td>$-\gamma/8$</td>
<td>0</td>
<td>$-\gamma/4$</td>
<td>0</td>
<td>$-\gamma/2$</td>
<td>0</td>
<td>$-\gamma$</td>
<td>$-\gamma$</td>
<td>1</td>
<td>$-\gamma$</td>
</tr>
</tbody>
</table>

where state 5 is the absorbing state that has an infinite loops to itself.
The optimal dual solution is

\begin{align*}
x^{*}_{01} &= 1, \\
x^{*}_{11} &= 1 + \gamma, \\
x^{*}_{21} &= 1 + \gamma + \gamma^2, \\
x^{*}_{32} &= 1 + \gamma + \gamma^2 + \gamma^3, \\
x^{*}_{41} &= 1, \\
x^{*}_{51} &= \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma}.
\end{align*}
Why Study the Dual Problems?

- The dual exists, and it is intrinsically connected to the primal problem.
- The dual problem typically forms an adversary problem with the same set of data; therefore, we can replace the $\min - \max$ optimization to $\min - \min$ optimization so that the two objectives are aligned together.
- They provide a certificate of the optimality to each other, where dual variables become the Lagrange multipliers of the primal.
- Algorithmically, one can solve one of them and solve the other as a by-product.
- They offer structural optimality insights: e.g., let $z(b)$ denote the minimal-value function of the primal problem when $b$ varies. Then
  \[ \nabla z(b) = y^* \]
  where $y^*$ is the optimal dual solution.