Optimality Conditions for General Constrained Optimization

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Chapter 11.1-8
Figure 1: Global and Local Minimizers of One-Variable Function in Interval $[a, e]$
A differentiable function $f$ of one variable defined on an interval $F = [a, e]$. If an interior-point $\bar{x}$ is a local/global minimizer, then $f'(\bar{x}) = 0$; if the left-end-point $\bar{x} = a$ is a local minimizer, then $f'(a) \geq 0$; if the right-end-point $\bar{x} = e$ is a local minimizer, then $f'(e) \leq 0$. first-order necessary condition (FONC) summarizes the three cases by a unified set of optimality/complementarity slackness conditions:

$$a \leq x \leq e, \quad f'(x) = y^a + y^e, \quad y^a \geq 0, \quad y^e \leq 0, \quad y^a(x - a) = 0, \quad y^e(x - e) = 0.$$ 

If $f'(\bar{x}) = 0$, then it is also necessary that $f(x)$ is locally convex at $\bar{x}$ for it being a local minimizer.

How to tell the function is locally convex at the solution? It is necessary $f''(\bar{x}) \geq 0$, which is called the second-order necessary condition (SONC), which we would explored further.

These conditions are still not, in general, sufficient. It does not distinguish between local minimizers, local maximizers, or saddle points.

If the second-order sufficient condition (SOSC): $f'''(\bar{x}) > 0$, is satisfied or the function is strictly locally convex, then $\bar{x}$ is a local minimizer.

Thus, if the function is convex everywhere, the first-order necessary condition is already sufficient.
Second-Order Optimality Condition for Unconstrained Optimization

Theorem 1 (First-Order Necessary Condition) Let \( f(x) \) be a \( C^1 \) function where \( x \in \mathbb{R}^n \). Then, if \( \bar{x} \) is a minimizer, it is necessarily \( \nabla f(\bar{x}) = 0 \).

Theorem 2 (Second-Order Necessary Condition) Let \( f(x) \) be a \( C^2 \) function where \( x \in \mathbb{R}^n \). Then, if \( \bar{x} \) is a minimizer, it is necessarily

\[
\nabla f(\bar{x}) = 0 \quad \text{and} \quad \nabla^2 f(\bar{x}) \succeq 0.
\]

Furthermore, if \( \nabla^2 f(\bar{x}) \succ 0 \), then the condition becomes sufficient.

The proofs would be based on 2nd-order Taylor’s expansion at \( \bar{x} \) such that if these conditions are not satisfied, then one would be find a descent-direction \( d \) and a small constant \( \bar{\alpha} > 0 \) such that \( f(\bar{x} + \alpha d) < f(\bar{x}) \), \( \forall 0 < \alpha \leq \bar{\alpha} \).

For example, if \( \nabla f(\bar{x}) = 0 \) and \( \nabla^2 f(\bar{x}) \preceq 0 \), the eigenvector of a negative eigenvalue of the Hessian would be a descent direction from \( \bar{x} \).

Again, they may still not be sufficient, e.g., \( f(x) = x^3 \).
General Constrained Optimization

\[
(GCO) \quad \min \; f(x) \\
\text{s.t.} \; h(x) = 0 \in \mathbb{R}^m, \\
\quad c(x) \geq 0 \in \mathbb{R}^p.
\]

We have dealt the cases when the feasible region is a convex polyhedron and/or the feasible can be represented by nonlinear convex cones intersect linear equality constraints.

We now study the case that the only assumption is that all functions are in $C^1$, and $C^2$ later, either convex or nonconvex.

We again establish optimality conditions to qualify/verify any local optimizers. These conditions give us qualitative structures of (local) optimizers and lead to quantitative algorithms to numerically find a local optimizer or an KKT solution.

The main proof idea is that if $\bar{x}$ is a local minimier of (GCO), then it must be a local minimizer of the problem where the constraints are linearized using the First-Order Taylor expansion.
Consider the (intersection) of Hypersurfaces (vs. Hyperplanes):

\[ \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{h}(\mathbf{x}) = \mathbf{0} \in \mathbb{R}^m, \ m \leq n \} \]

When functions \( h_i(\mathbf{x}) \)s are \( C^1 \) functions, we say the surface is smooth.

For a point \( \bar{\mathbf{x}} \) on the surface, we call it a regular point if \( \nabla \mathbf{h}(\bar{\mathbf{x}}) \) have rank \( m \) or the rows, or the gradient vector of each \( h_i(\cdot) \) at \( \bar{\mathbf{x}} \), are linearly independent. For example, \((0; 0)\) is not a regular point of \( \{ (x_1; x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - 1)^2 - 1 = 0, \ x_1^2 + (x_2 + 1)^2 - 1 = 0 \} \).

Based on the Implicit Function Theorem (Appendix A of the Text), if \( \bar{\mathbf{x}} \) is a regular point and \( m < n \), then for every \( \mathbf{d} \in \mathcal{T}_{\bar{\mathbf{x}}} = \{ \mathbf{z} : \nabla \mathbf{h}(\bar{\mathbf{x}}) \mathbf{z} = \mathbf{0} \} \) there exists a curve \( \mathbf{x}(t) \) on the hypersurface, parametrized by a scalar \( t \) in a sufficiently small interval \([-a \ a]\), such that

\[ \mathbf{h}(\mathbf{x}(t)) = \mathbf{0}, \ \mathbf{x}(0) = \bar{\mathbf{x}}, \ \dot{\mathbf{x}}(0) = \mathbf{d}. \]

\( \mathcal{T}_{\bar{\mathbf{x}}} \) is called the tangent-space or tangent-plane of the constraints at \( \bar{\mathbf{x}} \).
Figure 2: Tangent Plane on a Hypersurface at Point $x^*$
Lemma 1  

Let \( \bar{x} \) be a feasible solution and a regular point of the hypersurface of

\[
\{ x : h(x) = 0, c_i(x) = 0, i \in A_{\bar{x}} \}
\]

where active-constraint set \( A_{\bar{x}} = \{ i : c_i(\bar{x}) = 0 \} \). If \( \bar{x} \) is a (local) minimizer of (GCO), then there must be no \( d \) to satisfy linear constraints:

\[
\begin{align*}
\nabla f(\bar{x})d &< 0 \\
\nabla h(\bar{x})d &= 0 \in R^m, \\
\nabla c_i(\bar{x})d &\geq 0, \forall i \in A_{\bar{x}}.
\end{align*}
\]  

(1)

This lemma was proved when constraints are linear in which case \( d \) is a feasible direction, but needs more work otherwise since there is no feasible direction when constraints are nonlinear.

\( \bar{x} \) being a regular point is often referred as a Constraint Qualification condition.
Suppose we have a $\bar{d}$ satisfies all linear constraints. Then $\nabla f(\bar{x})\bar{d} < 0$ so that $\bar{d}$ is a descent-direction vector. Denote the active-constraint set at $\bar{d}$ among the linear inequalities in (1) by $A^d_{\bar{x}} (\subset A_{\bar{x}})$. Then, $\bar{x}$ remains a regular point of hypersurface of

$$\{ x : h(x) = 0, c_i(x) = 0, i \in A^d_{\bar{x}} \}.$$ 

Thus, there is a curve $x(t)$ such that

$$h(x(t)) = 0, \quad c_i(x(t)) = 0, \quad i \in A^d_{\bar{x}}, \quad x(0) = \bar{x}, \quad \dot{x}(0) = \bar{d},$$

for $t \in [0 \ a]$ of a sufficiently small positive constant $a$.

Also, $\nabla c_i(\bar{x})\bar{d} > 0$, $\forall i \notin A^d_{\bar{x}}$ but $i \in A_{\bar{x}}$; and $c_i(\bar{x}) > 0$, $\forall i \notin A_{\bar{x}}$. Then, from Taylor’s theorem, $c_i(x(t)) > 0$ for all $i \notin A^d_{\bar{x}}$ so that $x(t)$ is a feasible curve to the original (GCO) problem for $t \in [0 \ a]$. Thus, $\bar{x}$ must be also a local minimizer among all local solutions on the curve $x(t)$.

Let $\phi(t) = f(x(t))$. Then, $t = 0$ must be a local minimizer of $\phi(t)$ for $0 \leq t \leq a$ so that

$$0 \leq \phi'(0) = \nabla f(x(0))\dot{x}(0) = \nabla f(\bar{x})\bar{d} < 0, \quad \Rightarrow \text{ a contradiction.}$$
Theorem 3 (First-Order or KKT Optimality Condition) Let $\bar{x}$ be a (local) minimizer of (GCO) and it is a regular point of $\{x : h(x) = 0, c_i(x) = 0, i \in \mathcal{A}_x\}$. Then, for some multipliers $(\bar{y}, \bar{s} \geq 0)$

$$\nabla f(\bar{x}) = \bar{y}^T \nabla h(\bar{x}) + \bar{s}^T \nabla c(\bar{x})$$

(2)

and (complementarity slackness)

$$\bar{s}_i c_i(\bar{x}) = 0, \quad \forall i.$$

The proof is again based on the Alternative System Theory or Farkas Lemma. The complementarity slackness condition is from that $c_i(\bar{x}) = 0$ for all $i \in \mathcal{A}_x$, and for $i \notin \mathcal{A}_x$, we simply set $\bar{s}_i = 0$.

A solution who satisfies these conditions is called an KKT point or solution of (GCO) – any local minimizer $\bar{x}$, if it is also a regular point, must be an KKT solution; but the reverse may not be true.
It is more convenient to introduce the **Lagrangian Function** associated with generally constrained optimization:

\[
L(x, y, s) = f(x) - y^T h(x) - s^T c(x),
\]

where multipliers \( y \) of the equality constraints are “free” and \( s \geq 0 \) for the “greater or equal to” inequality constraints, so that the KKT condition (2) can be written as

\[
\nabla_x L(\bar{x}, \bar{y}, \bar{s}) = 0.
\]

Lagrangian Function can be viewed as a function aggregated the original objective function plus the penalized terms on constraint violations.

In theory, one can adjust the penalty multipliers \((y, s \geq 0)\) to repeatedly solve the following so-called **Lagrangian Relaxation Problem**: 

\[
(LRP) \quad \min_x \quad L(x, y, s).
\]
One condition for a local minimizer \( \bar{x} \) that must \textit{always} be an KKT solution is the constraint qualification: \( \bar{x} \) is a regular point of the constraints. Otherwise, a local minimizer may not be an KKT solution: Consider \( \bar{x} = (0; 0) \) of a convex nonlinearly-constrained problem

\[
\min x_1, \quad \text{s.t.} \quad x_1^2 + (x_2 - 1)^2 - 1 \leq 0, \quad x_1^2 + (x_2 + 1)^2 - 1 \leq 0.
\]

On the other hand, even the regular point condition does not hold, the KKT theorem may still true:

\[
\min x_2, \quad \text{s.t.} \quad x_1^2 + (x_2 - 1)^2 - 1 \leq 0, \quad x_1^2 + (x_2 + 1)^2 - 1 \leq 0,
\]

that is, \( \bar{x} = (0; 0) \) is an KKT solution of the latter problem.

Therefore, finding an KKT solution is a plausible way to find a local minimizer.
We now consider optimality conditions for problems having three types of inequalities:

\[
\begin{align*}
\text{(GCO)} & \\
\min & \quad f(x) \\
\text{s.t.} & \quad c_i(x) \ (\leq, =, \geq) 0, \ i = 1, \ldots, m, \quad \text{(Original Problem Constraints (OPC))}
\end{align*}
\]

For any feasible point \( x \) of (GCO) define the active constraint set by \( \mathcal{A}_x = \{i : c_i(x) = 0\} \).

Let \( \bar{x} \) be a local minimizer for (GCO) and \( \bar{x} \) is a regular point on the hypersurface of the active constraints.

Then there exist multipliers \( \bar{y} \) such that

\[
\begin{align*}
\nabla f(\bar{x}) &= \bar{y}^T \nabla c(\bar{x}) \quad \text{(Lagrangian Derivative Conditions (LDC))} \\
\bar{y}_i \ (\leq, \text{free}', \geq) \ &= \ 0, \ i = 1, \ldots, m, \quad \text{(Multiplier Sign Constraints (MSC))} \\
\bar{y}_i c_i(\bar{x}) &= 0, \quad \text{(Complementarity Slackness Conditions (CSC))}.
\end{align*}
\]

The complete First-Order KKT Conditions consist of these four parts!
Recall SOCP Relaxation of Sensor Network Localization

Given $a_k \in \mathbb{R}^2$ and Euclidean distances $d_k, \ k = 1, 2, 3$, find $x \in \mathbb{R}^2$ such that

$$\min_x \quad 0^T x,$$

$$\text{s.t.} \quad \|x - a_k\|^2 - d_k^2 \leq 0, \ k = 1, 2, 3,$$

$$L(x, y) = 0^T x - \sum_{k=1}^{3} y_k (\|x - a_k\|^2 - d_k^2),$$

$$0 = \sum_{k=1}^{3} y_k (x - a_k) \quad \text{(LDC)}$$

$$y_k \leq 0, \ k = 1, 2, 3, \quad \text{(MSC)}$$

$$y_k (\|x - a_k\|^2 - d_k^2) = 0. \quad \text{(CSC)}.$$
Arrow-Debreu’s Exchange Market with Linear Economy

Each trader $i$, equipped with a good bundle vector $w_i$, trade with others to maximize its individual utility function. The equilibrium price is an assignment of prices to goods so as when every producer sells his/her own good bundle and buys a maximal bundle of goods then the market clears. Thus, trader $i$’s optimization problem, for given prices $p_j$, $j \in G$, is

$$\begin{align*}
\text{maximize} & \quad u_i^T x_i := \sum_{j \in P} u_{ij} x_{ij} \\
\text{subject to} & \quad p^T x_i := \sum_{j \in P} p_j x_{ij} \leq p^T w_i, \\
& \quad x_{ij} \geq 0, \quad \forall j,
\end{align*}$$

Then, the equilibrium price vector is the one such that there are maximizers $x(p)_i$s

$$\sum_i x(p)_{ij} = \sum_i w_{ij}, \quad \forall j.$$
Example of Arrow-Debreu’s Model

Traders 1, 2 have good bundle

\[ w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Their optimization problems for given prices \( p_x, p_y \) are:

\[
\begin{align*}
\text{max} & \quad 2x_1 + y_1 \\
\text{s.t.} & \quad p_x \cdot x_1 + p_y \cdot y_1 \leq p_x, \\
& \quad x_1, y_1 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad 3x_2 + y_2 \\
\text{s.t.} & \quad p_x \cdot x_2 + p_y \cdot y_2 \leq p_y, \\
& \quad x_2, y_2 \geq 0.
\end{align*}
\]

One can normalize the prices \( p \) such that one of them equals 1. This would be one of the problems in HW2.
Equilibrium conditions of the Arrow-Debreu market

Similarly, the necessary and sufficient equilibrium conditions of the Arrow-Debreu market are

\[
p_j \geq u_{ij} \cdot \frac{p_i^T w_i}{x_i}, \quad \forall i, j,
\]

\[
\sum_i x_{ij} = \sum_i w_{ij} \quad \forall j,
\]

\[
p_j > 0, \quad x_i \geq 0, \quad \forall i, j;
\]

where the budget for trader \( i \) is replaced by \( p^T w_i \). Again, the nonlinear inequality can be rewritten as

\[
\log(u_i^T x_i) + \log(p_j) - \log(p^T w_i) \geq \log(u_{ij}), \quad \forall i, j, u_{ij} > 0.
\]

Let \( y_j = \log(p_j) \) or \( p_j = e^{y_j} \) for all \( j \). Then, these inequalities become

\[
\log(u_i^T x_i) + y_j - \log(\sum_j w_{ij} e^{y_j}) \geq \log(u_{ij}), \quad \forall i, j, u_{ij} > 0.
\]

Note that the function on the left is concave in \( x_i \) and \( y_j \).

Theorem 4  The equilibrium set of the Arrow-Debreu Market is convex in allocations and the logarithmic of prices.
Exchange Markets with Other Economies

Cobb-Douglas Utility:

\[ u_i(x_i) = \prod_{j \in G} x_{ij}^{u_{ij}}, \ x_{ij} \geq 0. \]

Leontief Utility:

\[ u_i(x_i) = \min_{j \in G} \left\{ \frac{x_{ij}}{u_{ij}}, \ x_{ij} \geq 0. \right\}. \]

Again, the equilibrium price vector is the one such that there are maximizers to clear the market.
Now in addition we assume all functions are in $C^2$, that is, twice continuously differentiable. Recall the tangent linear sub-space at $\bar{x}$:

$$T_{\bar{x}} := \{ z : \nabla h(\bar{x}) z = 0, \nabla c_i(\bar{x}) z = 0 \, \forall i \in A_{\bar{x}} \}. $$

**Theorem 5** Let $\bar{x}$ be a (local) minimizer of $(GCO)$ and a regular point of hypersurface

$$\{ x : h(x) = 0, c_i(x) = 0, i \in A_{\bar{x}} \},$$

and let $\bar{y}, \bar{s}$ denote Lagrange multipliers such that $(\bar{x}, \bar{y}, \bar{s})$ satisfies the (first-order) KKT conditions of $(GCO)$. Then, it is necessary to have

$$d^T \nabla_x^2 L(\bar{x}, \bar{y}, \bar{s}) d \geq 0 \quad \forall \, d \in T_{\bar{x}}. $$

The **Hessian** of the Lagrangian function need to be positive semidefinite on the tangent-space.
Proof

The proof reduces to one-dimensional case by considering the objective function \( \phi(t) = f(x(t)) \) for the feasible curve \( x(t) \) on the surface of ALL active constraints. Since 0 is a (local) minimizer of \( \phi(t) \) in an interval \([−a, a]\) for a sufficiently small \( a > 0 \), we must have \( \phi'(0) = 0 \) so that

\[
0 \leq \phi''(t)|_{t=0} = \dot{x}(0)^T \nabla^2 f(\bar{x}) \dot{x}(0) + \nabla f(\bar{x}) \ddot{x}(0) = d^T \nabla^2 f(\bar{x}) d + \nabla f(\bar{x}) \ddot{x}(0).
\]

Let all active constraints (including the equality ones) be \( h(x) = 0 \) and differentiating equations

\[
\bar{y}^T h(x(t)) = \sum_i \bar{y}_i h_i(x(t)) = 0
\]

twice, we obtain

\[
0 = \dot{x}(0)^T \left[ \sum_i \bar{y}_i \nabla^2 h_i(\bar{x}) \right] \dot{x}(0) + \bar{y}^T \nabla h(\bar{x}) \ddot{x}(0) = d^T \left[ \sum_i \bar{y}_i \nabla^2 h_i(\bar{x}) \right] d + \bar{y}^T \nabla h(\bar{x}) \ddot{x}(0).
\]

Let the second expression subtracted from the first one on both sides and use the FONC:

\[
0 \leq d^T \nabla^2 f(\bar{x}) d - d^T \left[ \sum_i \bar{y}_i \nabla^2 h_i(\bar{x}) \right] d + \nabla f(\bar{x}) \ddot{x}(0) - \bar{y}^T \nabla h(\bar{x}) \ddot{x}(0)
\]

\[
= d^T \nabla^2 f(\bar{x}) d - d^T \left[ \sum_i \bar{y}_i \nabla^2 h_i(\bar{x}) \right] d
\]

\[
= d^T \nabla^2_{\bar{x}} L(\bar{x}, \bar{y}, \bar{s}) d.
\]

Note that this inequality holds for every \( d \in T_{\bar{x}} \).
Theorem 6  Let $\bar{x}$ be a regular point of (GCO) with equality constraints only and let $\bar{y}$ be the Lagrange multipliers such that $(\bar{x}, \bar{y})$ satisfies the (first-order) KKT conditions of (GCO). Then, if in addition
\[ d^T \nabla^2_x L(\bar{x}, \bar{y}) d > 0 \quad \forall \ 0 \neq d \in T_{\bar{x}}, \]
then $\bar{x}$ is a local minimizer of (GCO).

See the proof in Chapter 11.5 of LY.

The SOSC for general (GCO) is proved in Chapter 11.8 of LY.
\[
\min (x_1)^2 + (x_2)^2 \quad \text{s.t.} \quad (x_1)^2/4 + (x_2)^2 - 1 = 0
\]
\[ L(x_1, x_2, y) = (x_1)^2 + (x_2)^2 - y(-\frac{(x_1)^2}{4} - (x_2)^2 + 1), \]

\[ \nabla_x L(x_1, x_2, y) = (2x_1(1 + y/4), 2x_2(1 + y)), \]

\[ \nabla^2_x L(x_1, x_2, y) = \begin{pmatrix} 2(1 + y/4) & 0 \\ 0 & 2(1 + y) \end{pmatrix} \]

\[ T_x := \{ (z_1, z_2) : (x_1/4)z_1 + x_2z_2 = 0 \}. \]

We see that there are two possible values for \( y \): either \(-4\) or \(-1\), which lead to total four KKT points:

\[ \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}. \]
Consider the first KKT point:

\[
\nabla^2_x L(2, 0, -4) = \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix}, \quad T_x = \{(z_1, z_2) : z_1 = 0\}
\]

Then the Hessian is not positive semidefinite on \(T_x\) since

\[
d^T \nabla^2_x L(2, 0, -4) d = -6d_2^2 \leq 0.
\]

Consider the third KKT point:

\[
\nabla^2_x L(0, 1, -1) = \begin{pmatrix} 3/2 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_x = \{(z_1, z_2) : z_2 = 0\}
\]

Then the Hessian is positive definite on \(T_x\) since

\[
d^T \nabla^2_x L(0, 0, -1) d = (3/2)d_1^2 > 0, \quad \forall 0 \neq d \in T_x.
\]

This would be sufficient for the third KKT solution to be a local minimizer.
Test Positive Semidefiniteness in a Subspace

In the second-order test, we typically like to know whether or not

\[ d^T Q d \geq 0, \quad \forall d, \text{ s.t. } A d = 0 \]

for a given symmetric matrix \( Q \) and a rectangle matrix \( A \). (In this case, the subspace is the null space of matrix \( A \).) This test itself might be a nonconvex optimization problem.

But it is known that \( d \) is in the null space of matrix \( A \) if and only if

\[ d = (I - A^T (AA^T)^{-1} A) u = P_A u \]

for some vector \( u \in R^n \), where \( P_A \) is called the projection matrix of \( A \). Thus, the test becomes whether or not

\[ u^T P_A Q P_A u \geq 0, \quad \forall u \in R^n, \]

that is, we just need to test positive semidefiniteness of \( P_A Q P_A \) as usual.
Another way is to apply SDP relaxation:

\[(SDP) \quad \min \quad Q \cdot D \]

\[
\text{s.t.} \quad A_i^T A_i \cdot D = 0; \quad \forall i \]

\[
D \succeq 0,
\]

where \(A_i\) is the \(i\)th row vector of \(A\). The objective value is bounded below by 0 if the dual has a feasible solution:

\[(SDD) \quad \min \quad 0^T y \]

\[
\text{s.t.} \quad Q - \sum_i y_i A_i^T A_i \succeq 0.
\]

Why?