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Nash and Correlated Equilibria

Abstract

In this document we will look at useful facts and definitions about Nash and correlated equilibria for two person games. More specifically we'll give a brief definition of both equilibria, and then we'll point out the relation between them.
We will next introduce a numerical approach aimed at calculating Nash Equilibria of a symmetric game we wish to explore in CME 334. A motivation for this approach will be given in a future document.

1. Game Theory Definitions and Properties

Let $1 \leq i \leq n$ and $1 \leq j \leq n$ be integers. $a_{i,j}^{(k)}$, the elements of A_k, are the payoff player k receives when he plays along the pure strategy i and the other player plays along the pure strategy j. This defines a two-player game in strategic form.

For a given finite set S, let $\Delta(S)$ be the set of all probability distributions on S. That is $\pi \in \Delta(S)$ iff $\pi \in [0,1]^{|S|}$ and $\sum_{i=1}^{|S|} \pi_i = 1$.

Definition 1 Nash Equilibrium for the game represented by (A_1, A_2).
Let C_1 and C_2 be the set of strategies available to players 1 and 2 respectively. We say that (π, ρ) is a Nash equilibrium iff $\pi \in \Delta(C_1)$, $\rho \in \Delta(C_2)$ and

1. $\forall \sigma_1 \in \Delta(C_1)$, $\pi^T A_1 \rho \geq \sigma_1^T A_1 \rho$
2. $\forall \sigma_2 \in \Delta(C_2)$, $\rho^T A_2 \pi \geq \sigma_2^T A_2 \pi$

In other words the expected payoff of player 1 is maximized for the probability distribution π given that player 2 uses probability distribution ρ (and similarly for player 2).

This is a quadratic problem. In order to transform this into a linear problem, we introduce the concept of *Correlated Equilibrium*

Definition 2 Let Γ be a nxn matrix whose elements are in $I\!R^+$ and such that $\sum_{i,j} \gamma_{ij} = 1$ (ie Γ is a probability distribution on the set of all possible pairs of strategies). We say that Γ is a correlated equilibria if the following conditions (called the *Strategic Incentive Constraints.*)are satisfied:

1. $\forall (i,k)$, $\sum_{j=1}^{n} \gamma_{i,j}.(a_{i,j}^{(1)} - a_{k,j}^{(1)}) \geq 0$; and
2. $\forall (j,k)$, $\sum_{i=1}^{n} \gamma_{i,j}.(a_{j,i}^{(2)} - a_{k,i}^{(2)}) \geq 0$

This defines a Linear Program, which we can solve efficiently.

The relationship between this two concepts is very direct. It is easy to prove that, given a Nash Equilibrium, if we define the matrix $\rho.\pi^T$, this matrix is a Correlated Equlibrium. From this definition it is obvious that the matrix is of rank-one. We can also prove that if the matrix M is a Correlated Equilibria and M is of rank-one, then M is a product distribution over the set of possible pairs of strategies, and that the factor distributions represent a Nash Equlibrium.

From the preceding observation, we can see that by relaxing the definition of Equilibrium from Nash to Correlated Equilibrium, we are able to compute efficiently the set of Correlated Equilibria. Now, to find a Nash Equilibrium we only need to find a point in the intersection of the set of Correlated Equilibria and the set of rank-one matrices. An additional remark is that one can look for a symmetric rank-one matrix when the matrices A_1 and A_2 are equal. We will focus on this particular case.

2. Numerical Approach to Explore

Given a correlated equilibrium, one could try to reduce the rank of the matrix to get as close as possible to a rank-one matrix. This problem is very complicated, so we will adopt a different approach. We will assume that we are in the rank-one matrices set and also that our resulting matrix is a probability distribution.

More precisely: Given a nxn matrix A with positive entries $a_{i,j}$, find a vector u in $\mathbb{R}^n$ such that

1. (i) $\forall i,\ u_i \geq 0$
2. (ii) $u^T e = 1$ where e is the vector with all entries equal to one
3. (iii) $\forall (i,j),\ u_i.u^T(A_i - A_j) \geq 0$ where A_i is the ith column of A

So far we have tried to define the distance of our current rank-one matrix to the polytope defined by (iii) (Call it P) as the absolute value of the sum of the violation of constraints.

More precisely: Given a vector u that satisfies both (i) and (ii) and a matrix A as above, we define the distance from u to the polytope P as

$$d_P(u) = \sum_{(i,j)\in V} u_i.u^T(A_j - A_i) \text{ with } V = \{(i,j)|u_i.u^T(A_i - A_j) < 0\}$$

My goal in the context of CME 334 project is to find an efficient numerical algorithm to solve the problem $min\{d_P(u)\}$ such that $u \geq 0$ and $u^T e = 1$.