Conic Linear Optimization and its Dual

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Conic Linear Programming (CLP)

The standard form conic linear programming problem is given below, which we will use throughout this book:

\[
\begin{align*}
(CLP) \quad & \text{minimize} \quad c \cdot x \\
& \text{subject to} \quad a_i \cdot x = b_i, \ i = 1, \ldots, m, \ x \in K,
\end{align*}
\]

where \( c, \ a_i \) in some inner product space.

With every (CLP), another linear program, denoted by (CLD), is called the dual of (CLP):

\[
\begin{align*}
(CL D) \quad & \text{maximize} \quad b^T y \\
& \text{subject to} \quad \sum_i^n y_i a_i + s = c, \ s \in K^*,
\end{align*}
\]

where \( y \in \mathbb{R}^m \) and \( s \in E \).
(SDP) \[ \inf \ C \cdot X \]
subject to \[ A_i \cdot X = b_i, \ i = 1, 2, \ldots, m, \ X \succeq 0, \]
where \( C, \ A_i \in \mathcal{S}^n \).

Then, the dual problem to (SDP) is:

(SDD) \[ \sup \ b^T y \]
subject to \[ \sum_i^m y_i A_i + S = C, \ S \succeq 0, \]
where \( y \in \mathbb{R}^m \) and \( S \in \mathcal{S}^n \).
Theorem 1 Let $\mathcal{F}_p$ and $\mathcal{F}_d$ denote the feasible regions of (CLP) and (CLD), respectively, and be non-empty. Then,

$$c \cdot x \geq b^T y \quad \text{where} \quad x \in \mathcal{F}_p, \ (y, s) \in \mathcal{F}_d.$$

$$c \cdot x - b^T y = (c - A^T y) \cdot x = s \cdot x \geq 0.$$  

This quantity is called the duality or complementarity gap.

Corollary 1 i) If (LP) or (LD) is feasible but unbounded (its objective value is unbounded) then the other is infeasible or has no feasible solution.

ii) If a pair of feasible solutions can be found to the primal and dual problems with an equal objective value, then they are both optimal.
**Strong Duality Theorem for LP**

**Theorem 2** We have

i) If (LP) or (LD) has no feasible solution, then the other is either unbounded or has no feasible solution.

ii) If (LP) or (LD) is feasible and bounded, then the other is feasible.

iii) If (LP) and (LD) both have feasible solutions then both problems have optimal solutions and the optimal objective values of the objective functions are equal, that is, optimal solutions for both (LP) and (LD) exist and there is no duality gap.

A case that neither (LP) nor (LD) is feasible:

\[ c = (-1; 0), \quad A = (0, -1), \quad b = 1. \]
Proof of LP Strong Duality Theorem

i) Let \( F_d \) be empty. Then, from the Farkas Lemma, there is a \( d \) such that

\[
    c^T d < 0 \quad \text{and} \quad A d = 0, \; d \geq 0.
\]

Thus, if (LP) is feasible, say \( x \in F_p \), then \( x + \alpha d \in F_p \) for any \( \alpha > 0 \). But its objective value \( c^T x + \alpha(c^T d) \) goes to \( -\infty \) as \( \alpha \) goes to \( \infty \).

ii) Let LP be feasible and bounded. Then, there is no \( d \) such that

\[
    c \cdot d < 0 \quad \text{and} \quad A d = 0, \; d \geq 0.
\]

Thus, from the Farkas Lemma, (LD) is feasible.

iii) We like to prove that there is a solution to

\[
    A x = b, \; A^T y + s = c, \; c^T x - b^T y \leq 0, \; (x, s) \geq 0,
\]

given that both (LP) and (LD) are feasible.
Suppose not, from Farkas’ lemma, we must have an infeasibility certificate 
\((x', \tau, y')\) such that

\[
Ax' - b\tau = 0, \quad A^Ty' - c\tau \leq 0, \quad (x'; \tau) \geq 0
\]

and

\[
b^Ty' - c^Tx' = 1
\]

If \(\tau > 0\), then we have

\[
0 \geq (-y')^T(Ax' - b\tau) + x'^T(A^Ty' - c\tau) = \tau(b^Ty' - c^Tx') = \tau
\]

which is a contradiction.

If \(\tau = 0\), then the weak duality theorem also leads to a contradiction, since it leads to one of (LP) and (LD) being unbounded.
Strong Duality Theorem for CLP?

Example 1:

\[
C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad K = S^2_+.
\]

Example 2:

\[
C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\]

and

\[
b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}; \quad K = S^3_+.
\]
The key is that, in these examples, the Farkas lemma does not hold.

**Theorem 3** We have

i) Let (CLP) or (CLD) be infeasible, and furthermore the other be feasible and have an interior. Then the other is unbounded.

ii) Let (CLP) and (CLD) be both feasible, and furthermore one of them have an interior. Then there is no duality gap between (CLP) and (CLD).

iii) Let (CLP) and (CLD) be both feasible and have interior. Then, both have optimal solutions with no duality gap.
Proof of CLP Strong Duality Theorem

i) Suppose $\mathcal{F}_d$ is empty and $\mathcal{F}_p$ be feasible and have an interior. Then, we have $\bar{x} \in \text{int} K$ and $\bar{\tau} > 0$ such that $A\bar{x} - b\bar{\tau} = 0$, $(\bar{x}, \bar{\tau}) \in \text{int}(K \times R_+)$. Then, for any $z^*$, we have an alternative system pair

$$Ax - b\tau = 0, \ c \cdot x - z^*\tau < 0, \ (x, \tau) \in K \times R_+,$$

and

$$A^T y + s = c, \ -b^T y + s = -z^*, \ (s, s) \in K^* \times R_+.$$

But the latter is infeasible, so that the formal has a feasible solution for any $z^*$. At such an solution, if $\tau > 0$, we have $c \cdot (x/\tau) < z^*$; if $\tau = 0$, we have $\hat{x} + \alpha x$, where $\hat{x}$ is any feasible solution for (CLP), being feasible for (CLP) and its objective value goes to $-\infty$ as $\alpha$ goes to $\infty$.

ii) Let $\mathcal{F}_p$ be feasible and have an interior, and let $z^*$ be its infimum. Again, we
have an alternative system pair

\[ Ax - b\tau = 0, \ c \cdot x - z^*\tau < 0, \ (x, \tau) \in K \times R_+, \]

and

\[ A^T y + s = c, \ -b^T y + s = -z^*, \ (s, s) \in K^* \times R_+. \]

But the former is infeasible, so that we have a solution for the latter. From the Weak Duality theorem, we must have \( s = 0 \), that is, we have a solution \((y, s)\) such that

\[ A^T y + s = c, \ b^T y = z^*, \ s \in K^*. \]

iii) We only need to prove that there exist a solution \( x \in F_p \) such that \( c \cdot x = z^* \), that is, the infimum of (CLP) is attainable. But this is just the other side of the proof given that \( F_d \) is feasible and has an interior, and \( z^* \) is also the supremum of (CLD).
SDP Example with Zero-Duality Gap but not Attainable

\[ C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad b_1 = 2. \]

The primal has an interior, but the dual does not.
LP Optimality Conditions

\[
\begin{align*}
\mathbf{c}^T \mathbf{x} - \mathbf{b}^T \mathbf{y} &= 0 \\
(\mathbf{x}, \mathbf{y}, \mathbf{s}) &\in (\mathbb{R}^+_n, \mathbb{R}^m, \mathbb{R}^+_n) : \\
A \mathbf{x} &= \mathbf{b} \\
-A^T \mathbf{y} - \mathbf{s} &= -\mathbf{c}
\end{align*}
\]

which is a system of linear inequalities and equations. Now it is easy to verify whether or not a pair \((\mathbf{x}, \mathbf{y}, \mathbf{s})\) is optimal.

Since both \(\mathbf{x}\) and \(\mathbf{s}\) are nonnegative, \(\mathbf{x}^T \mathbf{s} = 0\) implies that \(x_j s_j = 0\) for all \(j = 1, \ldots, n\), where we say \(\mathbf{x}\) and \(\mathbf{s}\) are complementary to each other.

\[
\begin{align*}
\mathbf{x} \cdot \mathbf{s} &= 0 \\
A \mathbf{x} &= \mathbf{b} \\
-A^T \mathbf{y} - \mathbf{s} &= -\mathbf{c}
\end{align*}
\]

This system has total \(2n + m\) unknowns and \(2n + m\) equations including \(n\) nonlinear equations.
Solution Support for LP

Let $x^*$ and $s^*$ be optimal solutions with zero duality gap. Then

$$|\text{supp}(x^*)| + |\text{supp}(s^*)| \leq n.$$ 

There are $x^*$ and $s^*$ such that the support sizes of $x^*$ and $s^*$ are maximal, respectively.

There are $x^*$ and $s^*$ such that the support size of $x^*$ and $s^*$ are minimal, respectively.

If there is $s^*$ such that $|\text{supp}(s^*)| \geq n - d$, then the support size for $x^*$ is $d$. 
**LP strict complementarity theorem**

**Theorem 4** If (LP) and (LD) are both feasible, then there exists a pair of strictly complementary solutions $\mathbf{x}^* \in \mathcal{F}_p$ and $(\mathbf{y}^*, s^*) \in \mathcal{F}_d$ such that

$$\mathbf{x}^* \cdot s^* = 0 \quad \text{and} \quad |\text{supp}(\mathbf{x}^*)| + |\text{supp}(s^*)| = n.$$ 

Moreover, the supports

$$P^* = \{ j : x^*_j > 0 \} \quad \text{and} \quad Z^* = \{ j : s^*_j > 0 \}$$

are invariant for all strictly complementary solution pairs.

Given (LP) or (LD), the pair of $P^*$ and $Z^*$ is called the strict complementarity partition. $\{ \mathbf{x} : A_{P^*}\mathbf{x}_{P^*} = \mathbf{b}, \mathbf{x}_{P^*} \geq 0, \mathbf{x}_{Z^*} = \mathbf{0} \}$ is called the primal optimal face, and $\{ \mathbf{y} : c_{Z^*} - A_{Z^*}^T\mathbf{y} \geq 0, c_{P^*} - A_{P^*}^T\mathbf{y} = 0 \}$ is called the dual optimal face.
Consider the primal LP problem:

minimize \[ 2x_1 + x_2 + x_3 \]
subject to \[ x_1 + x_2 + x_3 = 1, \]
\[ (x_1, x_2, x_3) \geq 0, \]

where

\[ P^* = \{2, 3\} \quad \text{and} \quad Z^* = \{1\}. \]
Optimality Conditions for SDP with zero duality gap

\[ C \bullet X - b^T y = 0 \]
\[ AX = b \]
\[ -A^T y - S = -C \]
\[ X, S \succeq 0, \]

or

\[ XS = 0 \]
\[ AX = b \]
\[ -A^T y - S = -C \]
\[ X, S \succeq 0 \]
Solution Rank for SDP

Let $X^*$ and $S^*$ be optimal solutions with zero duality gap. Then

$$\operatorname{rank}(X^*) + \operatorname{rank}(S^*) \leq n.$$ 

There are $X^*$ and $S^*$ such that the ranks of $X^*$ and $S^*$ are maximal, respectively.

There are $X^*$ and $S^*$ such that the ranks of $X^*$ and $S^*$ are minimal, respectively.

If there is $S^*$ such that $\operatorname{rank}(S^*) \geq n - d$, then the maximal rank of $X^*$ is bounded by $d$. 
SDP strict complementarity?

Given a pair of SDP and (SDD) where the complementarity solution exist, is there a solution pair such that

\[ \text{rank}(X^*) + \text{rank}(S^*) = n? \]

\[
C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad A_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\]

and

\[
b = \begin{pmatrix} 0 \\ 0 \end{pmatrix};\quad K = S_+^3.
\]
Uniqueness Theorem for CLP: LP case

Given an optimal solution $x^*$, how to certify the uniqueness of $x^*$? We first consider the LP case.

**Theorem 5** An LP optimal solution $x^*$ is unique if and only if the size of $\text{supp}(x^*)$ is maximal among all optimal solutions and the columns of $A_{\text{supp}(x^*)}$ are linear independent.

It is easy to see both conditions are necessary, since otherwise, one can find an optimal solution with a different support size. To see sufficiency, suppose there is another optimal solution $y^*$ such that $x^* - y^* \neq 0$. We must have $\text{supp}(y^*) \subset \text{supp}(x^*)$. Then we see

$$0 = Ax^* - Ay^* = A(x^* - y^*) = A_{\text{supp}(x^*)}(x^* - y^*)_{\text{supp}(x^*)}$$

which implies that columns of $A_{\text{supp}(x^*)}$ are linearly dependent.
Uniqueness Theorem for CLP: SDP case

Given an SDP optimal and complementary solution $X^*$, how to certify the uniqueness of $X^*$?

**Theorem 6** An SDP optimal and complementary solution $X^*$ is unique if and only if the rank of $X^*$ is maximal among all optimal solutions and $V^*A_i(V^*)^T$, $i = 1, \ldots, m$, are linearly independent, where $X^* = (V^*)^TV^*$, $V^* \in \mathcal{R}^{r \times n}$, and $r$ is the rank of $X^*$.

It is easy to see why the rank of $X^*$ being maximal is necessary.

Note that for any optimal dual slack matrix $S^*$, we have $S^* \bullet (V^*)^TV^* = 0$ which implies that $S^*(V^*)^T = 0$. Consider any matrix

$$X = (V^*)^TV^*$$

where $U \in S_+^r$ and

$$b_i = A_i \bullet (V^*)^TU = V^*A_i(V^*)^T \bullet U, \ i = 1, \ldots, m.$$
One can see that $X$ remains an optimal SDP solutions for any such $U \in S^r_+$, since it makes $X$ feasible and remain complementary to any optimal dual slack matrix. If $V^* A_i (V^*)^T$, $i = 1, \ldots, m$, are not linearly independent, then one can find

$$V^* A_i (V^*)^T \bullet W = 0, \ i = 1, \ldots, m, \ 0 \neq W \in S^r.$$

Now consider

$$X(\alpha) = (V^*)^T (I + \alpha \cdot W)V^*,$$

and then we can choose $\alpha \neq 0$ such that $X(\alpha) \succeq 0$ is another optimal solution.

To see sufficiency, suppose there there is another optimal solution $Y^*$ such that $X^* - Y^* \neq 0$. We must have $Y^* = (V^*)^T U V^*$ for some $I \neq U \in S^r_+$. Then we see

$$V^* A_i (V^*)^T \bullet (I - U) = 0, \ i = 1, \ldots, m,$$

contradicts that they are linear independent.