Conic Linear Optimization Review

Yinyu Ye
Department of Management Science and Engineering
Stanford University
Stanford, CA 94305, U.S.A.

http://www.stanford.edu/~yyye
Convex Cone Examples

- The $n$-dimensional non-negative orthant, $\mathcal{R}^n_+ = \{ x \in \mathcal{R}^n : x \geq 0 \}$, is a convex cone; and it’s self dual.

- The set of all positive semi-definite symmetric matrices in $\mathcal{S}^n$, $\mathcal{S}^n_+$, is a convex cone, called the positive semi-definite matrix cone; and it’s self dual.

- The set $\{ x \in \mathcal{R}^n : x_1 \geq \| x_{-1} \| \}$, $\mathcal{N}_2^n$, is a convex cone in $\mathcal{R}^n$ called the second-order (norm) cone; and it’s self dual.
For any convex function \( c(x) \) over a convex set, consider convex cone

\[
C = \{(t; x) : t > 0, \, tc(x/t) \leq 0, \, x \in \mathcal{R}^n \}.
\]

\( c(x) \):

(a) \( p \)-order (norm) cone: \( c(x) = \|x\|_p - 1, \, p = 1, 2, \ldots \);

(b) Ellipsoidal cone: \( c(x) = x^T Q x - 1 \), where \( Q \) is a (symmetric) positive definite matrix.

(c) Exponential cone \( c(x) = \sum_{i=1}^{n} e^{ax_i} - 1 \).
The most important theorem about the convex set is the following **separating hyperplane** theorem.

**Theorem 1** *(Separating hyperplane theorem)* Let $C \subset \mathcal{E}$, where $\mathcal{E}$ is either $\mathbb{R}^n$ or $S^n$, be a closed convex set and let $b$ be a point exterior to $C$. Then there is a vector $a \in \mathcal{E}$ such that

$$a \cdot b > \sup_{x \in C} a \cdot x$$

where $a$ is the norm direction of the hyperplane.
Farkas Lemma for General Convex Cone

Theorem 2  Consider system \( \{ x : \ A x = b, \ x \in K \} \) for a (closed) convex cone \( K \). Suppose that there exists vector \( \bar{y} \) such that \( -A^T \bar{y} \in \text{int} \ K^* \). Then,

- Set \( C := \{ A x : \ x \in K \} \) is a closed convex set.
- The system \( \{ x : \ A x = b, \ x \in K \} \) has a feasible solution \( x \) if and only if that \( \{ y : \ -A^T y \in K^*, \ b^T y > 0, \ (b^T y = 1) \} \) has no feasible solution.

Theorem 3  Consider system \( \{(y, s) : \ A^T y + s = c, \ s \in K \} \) for a (closed) convex cone \( K \). Suppose that there exists vector \( \bar{x} \) such that \( A \bar{x} = 0, \ \bar{x} \in \text{int} \ K^* \). Then,

- Set \( C := \{ A^T y + s : \ s \in K \} \) is a closed convex set.
- The system \( \{ x : \ A x = 0, \ c \cdot x = -1, \ x \in K^* \} \) has a feasible solution \( x \) if and only if that \( \{(y, s) : \ A^T y + s = c, \ s \in K \} \) has no feasible solution.
Conic Linear Programming/Optimization

\[ (CLP) \quad \inf \quad C \cdot X \]
subject to \[ A_i \cdot X = b_i, \quad i = 1, 2, \ldots, m, \quad X \in \mathcal{K}. \]

The dual problem to (SDP) can be written as:

\[ (CLD) \quad \sup \quad b^T y \]
subject to \[ \sum_i^m y_i A_i + S = C, \quad S \in \mathcal{K}^*, \]

where \( y \in \mathcal{R}^m \).
### Rules to Construct the Dual

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<th>Min model</th>
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**Note:**
- \(K^*\) represents the dual cone of \(K\).
- \(\bar{K}\) represents the negative cone of \(K\).
Weak Duality Theorems for CLP

**Theorem 4** (Weak duality theorem in CLP) Let $\mathcal{F}_p$ and $\mathcal{F}_d$, the feasible sets for the primal and dual, be non-empty. Then,

$$C \cdot X \geq b^T y \text{ where } X \in \mathcal{F}_p, (y, S) \in \mathcal{F}_d.$$

The weak duality theorem is identical to that of (LP) and (LD).

**Corollary 1** Let $\mathcal{F}_p$ and $\mathcal{F}_d$, the feasible sets for the primal and dual, be non-empty. If one has an interior, then other is bounded.
The key is that, in these examples, the Farkas lemma does not hold.

**Theorem 5**  

i) Let (CLP) or (CLD) be infeasible, and furthermore the other be feasible and have an interior. Then the other is unbounded.

ii) Let (CLP) and (CLD) be both feasible, and furthermore one of them have an interior. Then there is no duality gap between (CLP) and (CLD).

iii) Let (CLP) and (CLD) be both feasible and have interior. Then, both have optimal solutions with no duality gap.
Let $X^*$ and $S^*$ be optimal solutions with zero duality gap. Then
\[ \text{rank}(X^*) + \text{rank}(S^*) \leq n. \]

There are $X^*$ and $S^*$ such that the ranks of $X^*$ and $S^*$ are maximal, respectively.

There are $X^*$ and $S^*$ such that the ranks of $X^*$ and $S^*$ are minimal, respectively.

If there is $S^*$ such that $\text{rank}(S^*) \geq n - d$, then the maximal rank of $X^*$ is bounded by $d$. 
Uniqueness Theorem for CLP: SDP case

Given an SDP optimal and complementary solution $X^*$, how to certify the uniqueness of $X^*$?

**Theorem 6** An SDP optimal and complementary solution $X^*$ is unique if and only if the rank of $X^*$ is maximal among all optimal solutions and $V^* A_i (V^*)^T$, $i = 1, \ldots, m$, are linearly independent, where $X^* = (V^*)^T V^*$, $V^* \in \mathbb{R}^{r \times n}$, and $r$ is the rank of $X^*$. 
**Bound on Rank**

**Theorem 7** (Barvinok (2001), Pataki (1999), and Alfakih/Wolkowicz (1999)) If there is a minimizer for (SDP), then there is a minimizer of (SDP) whose rank $r$ satisfying

$$
\frac{r(r + 1)}{2} \leq m.
$$

**How Sharp is the Rank Bound?** The rank bound is sharp: consider $n = 4$ and the SDP problem:

$$(e_i - e_j)(e_i - e_j)^T \bullet X = 1, \forall i < j = 1, 2, 3, 4,$$

$$X \succeq 0,$$
SDP Rank Reduction I: Eigenvalue Based Reduction

Let $X$ be an SDP solution with rank $r$ and

$$X = \sum_{i=1}^{r} \lambda_i v_i v_i^T$$

where

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$  

Then, let

$$\hat{X} = \sum_{i=1}^{d} \lambda_i v_i v_i^T$$
SDP Rank Reduction II: Randomized Reduction

Let $X$ be an SDP solution with rank $r$ and

$$X = V V^T$$

where $V \in \mathbb{R}^{n \times r}$ is factorization matrix of $X$.

Then, let random matrix

$$R = \sum_{i}^{d} \xi_i \xi_i^T, \quad \xi_i \in N(0, \frac{1}{d}I); \quad \text{or} \quad \xi_i \in \text{Binary}(0, \frac{1}{d}I)$$

(each entry either 1 or $-1$ in the binary case), and

$$\hat{X} = V R V^T.$$

Note that

$$E[\hat{X}] = V E[R] V^T = V V^T = X.$$
More SDP Rank Reduction III: \([-1, 1]\) Randomized Reduction

Let \(X\) be an SDP solution with rank \(r\) and

\[ X = VV^T. \]

Then, let random vector

\[ u \in N(0, I) \quad \text{and} \quad \hat{x} = \text{Sign}(Vu) \]

where

\[
\text{Sign}(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
-1 & \text{otherwise.}
\end{cases}
\]

Note that \(Vu \in N(0, X)\). It was proved by Sheppard (1900):

\[
E[\hat{x}_i \hat{x}_j] = \frac{2}{\pi} \arcsin(\bar{X}_{ij}), \quad i, j = 1, 2, \ldots, n.
\]
Construct a suitable objective for the SDP feasible set

Minimize \( R \cdot X \)

Subject to \( A_i \cdot X = b_i, \ i = 1, \ldots, m, \)

\( X \succeq 0. \)

The selection of matrix \( R \) is problem dependent.
SDP Rank Reduction V: Adding a Nonlinear Objective

Construct a suitable objective for the SDP solution set

Minimize \[ F_p(X) \]
Subject to \[ A_i \cdot X = b_i, \quad i = 1, \ldots, m, \]
\[ X \succeq 0. \]

Typically, \( F_p(X) \) is the Schatten \( p \)quasinnorm function.
Oplus of Cones

\( \mathcal{K} \) can be the plus of mixed cones:

\[
\mathcal{K} = K_1 \oplus K_2 \ldots \oplus K_p
\]

and it dual is

\[
\mathcal{K}^* = K_1^* \oplus K_2^* \ldots \oplus K_p^*.
\]
Example of Mixed Cones I

\( (SDP) \quad \inf \quad C \cdot X \)

subject to \( A_i \cdot X \leq b_i, i = 1, 2, \ldots, m, \ X \in \mathcal{K}. \)

\( (SDP) \quad \inf \quad C \cdot X \)

subject to \( A_i \cdot X + x_i \leq b_i, i = 1, 2, \ldots, m, \ X \in \mathcal{K}, \ x \geq 0. \)

\[ \mathcal{K} = S^n_+ \bigotimes \mathcal{R}^n_+ \]
Example of Mixed Cones II

minimize \[ \sum_i \|x - a_i\|_p \]
subject to \[ x \geq 0 \]

minimize \[ \sum_i t_i \]
subject to \[ x - u_i = a_i, \ (t_i; u_i) \in P, \ \forall i, \ x \geq 0 \]
Dual of the SOC Example

Let

$$A = \begin{pmatrix} I & 0 & -I & 0 & 0 & \ldots & 0 & 0 \\ I & 0 & 0 & 0 & -I & \ldots & 0 & 0 \\ \vdots \\ I & 0 & 0 & 0 & 0 & \ldots & 0 & -I \end{pmatrix}, \quad b = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}.$$ 

The dual problem can be written as:

$$\text{maximize} \quad \sum_i a_i^T y_i$$

$$\text{subject to} \quad \sum_i y_i \leq 0$$

$$(1; y_i) \in Q, \forall i.$$
SDP Algorithms

- Barrier and Potential Functions
- Interior-Point Algorithms
- Steepest Descent Methods
- Alternating Direction Methods with Multipliers
SDP Applications

- SDP Relaxation of QCQP problems
- SDP for Sensor Network Localization
- SDP for Robust Optimization