A few words about Cournot

Antoine-Augustin Cournot (1801–1877), a French economist and mathematician, is noted for his work on the analysis partial market equilibria. His analysis is based on the assumption that participants in the exchange process are either producers or merchants whose goal is the maximization of monetary profit. He did not introduce the concept of utility. He did, however, make important contributions to the discussion of supply and demand functions as well as to the establishment of equilibria under conditions of monopoly, duopoly, and perfect competition. He is the author of a book called Recherches sur les Principes Mathématiques de la Théorie des Richesses published in 1838. This book is available in English translation under the title Researches into the Mathematical Principles of the Theory of Wealth.

Source: Encyclopædia Britannica
Consider a set $\mathcal{F} = \{1, \ldots, M\}$ whose elements are firms $f$ in a spatially separated market modeled by a network $(\mathcal{N}, \mathcal{A})$.

The firms are assumed to produce the same commodity.

Firm $f$ has plants at a set of nodes $\mathcal{N}_f \subset \mathcal{N}$.

Let $x_{fa}$ be the amount of flow controlled by $f$ on link $a \in \mathcal{A}$.

Let $s_{fi}$ be the amount of the commodity produced by firm $f$ at node $i \in \mathcal{N}_f$.  [Formally, set $s_{fi} = 0$ if $i \in \mathcal{N} \setminus \mathcal{N}_f$.]

Let $C_{fi}(s_{fi})$ be the cost to $f$ of producing $s_{fi}$ units of the commodity at $i$.  [Formally, set $C_{fi}(s_{fi}) = 0$ if $i \in \mathcal{N} \setminus \mathcal{N}_f$.]

Let $\text{CAP}_{fi}$ be the [known, constant] capacity of firm $f$ to produce the commodity at $i$.

Let $d_{fj}$ be the amount of the commodity delivered by firm $f$ to node $j \in \mathcal{N}$. 
Let $Q_j = \sum_{f=1}^{M} d_{fj}$ be the total delivered to node $j$ by all firms.

Think of $Q_j$ as the demand at $j$.

Let $p_j(Q_j)$ be the unit price of the commodity at $j$ given that $Q_j$ units are delivered there.

Then $p_j(\cdot)$ is the inverse demand function at $j$.

Let $c_{fa}(x_{fa})$ be the cost to $f$ of shipping $x_{fa}$ units on link $a \in \mathcal{A}$.

Let $\mathcal{A}_i^+ = \{a \in \mathcal{A} : a = (i, j), j \in \mathcal{N}\}$.

Let $\mathcal{A}_i^- = \{a \in \mathcal{A} : a = (j, i), j \in \mathcal{N}\}$.

Firm $f$ needs to determine the variables

$$\{d_{fj} : j \in \mathcal{N}\}, \quad \{s_{fi} : i \in \mathcal{N}_f\}, \quad \{x_{fa} : a \in \mathcal{A}\}.$$ 

Let $x^f = (d_f, s_f, x_f)$. 
Let $K_f$ be the set of feasible production/distribution patterns of firm $f$.

$$K_f=\{x^f \geq 0 : s_{fi} \leq \text{CAP}_{fi}, \forall i \in \mathcal{N}_f,$$
$$d_{fi} + \sum_{a \in \mathcal{A}_i^+} x_{fa} = s_{fi} + \sum_{a \in \mathcal{A}_i^-} x_{fa}, \forall i \in \mathcal{N}_f,$$
$$d_{fi} + \sum_{a \in \mathcal{A}_i^+} x_{fa} = \sum_{a \in \mathcal{A}_i^-} x_{fa}, \forall i \in \mathcal{N}_f\}$$

Let $x = (x^1, \ldots, x^M)$, the direct sum of the vectors $x^f$.

The profit function for firm $f$ is its revenue minus its costs:

$$\theta_f(x) = \sum_{j \in \mathcal{N}} d_{fj}p_j \left( \sum_{g=1}^{M} d_{gj} \right) - \sum_{i \in \mathcal{N}_f} C_{fi}(s_{fi}) - \sum_{a \in \mathcal{A}} x_{fa}c_a(x_{fa}).$$
Note that

\[ \theta_f(x) = \sum_{j \in N} d_{fj} p_j(Q_j) - \sum_{i \in N_f} C_{fi}(s_{fi}) - \sum_{a \in A} x_{fa} c_a(x_{fa}). \]

This depends on the decisions of the other firms as well as firm \( f \).

Firm \( f \) has the optimization problem

maximize \( \theta_f(x) \)
subject to \( x^f \in K_f \).

**The equilibrium problem:** Find \( x \equiv (x^1, \ldots, x^M) \) such that for \( f = 1, \ldots, M \)

\[ x^f \in \text{arg max}\{ \theta_f(x) : x^f \in K_f \} \]

Such a vector would be called a *Cournot equilibrium*. 
Interpretation as a VI problem

To cast the above equilibrium problem as a variational inequality problem, we need to have each $\theta_f$ be concave in $x^f$. We also want it to be twice continuously differentiable.

Let $F$ be the mapping of the variational inequality formulation of the Nash-Cournot problem.

Then $F(x) = \bigoplus_{f=1}^{M} (-\nabla_{x^f} \theta_f(x))$.

Note that $x = \bigoplus_{f=1}^{M} x^f = \bigoplus_{f=1}^{M} (d_f, s_f, x_f)$.

The matrix $\nabla F(x)$ is a large, sparse, and has block structure.
If the inverse demand functions $p_j(Q_j)$ are linear, their Hessian matrices are zero, hence Jacobian matrix $\nabla \mathbf{F}(\mathbf{x})$ turns out to be symmetric. This implies $\mathbf{F}(\mathbf{x})$ is the gradient of a differentiable function.

When the inverse demand functions $p_j(Q_j)$ are nonlinear, it can turn out that the inverse demand function is not the gradient of a differentiable function. In short, it is not integrable and this means that there is no equivalent optimization problem.
An oligopolistic electricity model

There are three main aspects of electric power modeling: generation, transmission, and distribution.

Suppose that in some region there are a few firms that generate electric power and that transmission is controlled through capacity constraints.

In this model, a firm may have multiple generation plants at each of its sites $i \in \mathcal{N}_f$. This model does not include real power losses and does not consider investment in power generation capacity. (Both are rather important issues.)

Some further notation:

\begin{align*}
    G_{fi} &= \text{set of generation plants owned by firm } f \text{ at node } i \in \mathcal{N}_f \\
    \text{CAP}_{fih} &= \text{generation capacity at plant } h \in G_{fi} \\
    \text{CAP}_a &= \text{transmission capacity on link } a \\
    y_{fih} &= \text{amount produced at plant } h \in G_{fi} \\
    \rho_a &= \text{transmission price on link } a
\end{align*}
Firm $f$ needs to determine the variables

$$\{d_{fj}: j \in \mathcal{N}\}, \quad \{y_{fih}: i \in \mathcal{N}_f, \ h \in G_{fi}\},$$

and

$$z^f \equiv \{x_{fa}: a \in \mathcal{A}\}$$

As before, we have $x^f$ and $x = \bigoplus_{f \in \mathcal{F}} x^f$. Define

$$\tilde{x}^f = \bigoplus_{g \in \mathcal{F}\setminus\{f\}} x^g$$

and

$$z = \bigoplus_{f \in \mathcal{F}} z^f$$

Assume that the transmission prices $\rho_a(z)$ are given.
The constraint set for firm \( f \) (given \( \bar{x}^f \)) is:

\[
K_f(\bar{x}^f) \equiv \{x^f \geq 0 : y_{fi} \leq \text{CAP}_{fi}, \; \forall h \in G_{fk}, \; \forall i \in N_f \\
d_{fi} + \sum_{a \in A_{i}^+} x_{fa} = \sum_{h \in G_{fi}} y_{fi} + \sum_{a \in A_{i}^-} x_{fa}, \; \forall i \in N_f \\
d_{fi} + \sum_{a \in A_{i}^+} x_{fa} = \sum_{a \in A_{i}^-} x_{fa}, \; \forall i \in N \setminus N_f \\
\sum_{f' \in F} x_{f'a} \leq \text{CAP}_{a}, \; a \in A \}
\]

Note that these constraints are linear.

The last of these conditions is called the link capacity constraint and must be satisfied by all firms.
The profit function for firm $f$:

$$
\theta_f(x) = \sum_{j \in N} d_{fj} p_j \left( \sum_{g=1}^{M} d_{gj} \right) - \sum_{i \in N_f} \sum_{h \in G_{fi}} C_{fih}(y_{fih}) - \sum_{a \in A} x_f a \rho_a.
$$

$C_{fih}$ is the cost of generation to firm $f$ at site $i$ and plant $h$.

The profit maximization problem for firm $f$ (given $\tilde{x}^f$ and $\rho_a$ for all $a \in A$):

maximize $\theta_f(x^f)$

subject to $x^f \in K_f(\tilde{x}^f)$

The (Nash-Cournot) equilibrium problem in this case:

Find a vector $x = (x^1, \ldots, x^M)$ such that $x^f$ is optimal for firm $f$ and $\rho_a = \rho_a(z)$ for all $a \in A$. 
This would seem to be a QVI problem, but it can be formulated as a conventional VI problem. To that end, define

\[
\tilde{K}_f \equiv \{ x^f \geq 0 : y_{fih} \leq \text{CAP}_{fih}, \forall h \in G_{fk}, \forall i \in \mathcal{N}_f \}
\]

\[
d_{fi} + \sum_{a \in A_i^+} x_{fa} = \sum_{h \in G_{fi}} y_{fih} + \sum_{a \in A_i^-} x_{fa}, \quad \forall i \in \mathcal{N}_f
\]

\[
d_{fi} + \sum_{a \in A_i^+} x_{fa} = \sum_{a \in A_i^-} x_{fa}, \quad \forall i \in \mathcal{N} \setminus \mathcal{N}_f
\]

The set \( \tilde{K}_f \) does not depend on \( x \). Define

\[
K_f(x) = \bigcap_{a \in A} \{ x^{f'} : \sum_{f' \in \mathcal{F}} x_{f'a} \leq \text{CAP}_a \} \cap \tilde{K}_f
\]

\( \Omega \equiv \{ x : x \text{ satisfies all link capacity constraints} \} \)

\( d \equiv (d_{fi} : f \in \mathcal{F}, i \in \mathcal{N}) \)

\( y \equiv (y_{fih} : f \in \mathcal{F}, i \in \mathcal{N}, h \in G_{fi}) \)
Define the marginal return and marginal costs

$$MR_{fi}(d) = \frac{\partial \theta_f(x)}{\partial d_{fi}}$$ and $$MC_{fih} = \frac{dC_{fih}(y_{fih})}{dy_{fih}}$$

and the vector-valued function

$$F(d, y, z) \equiv \begin{bmatrix} -MR_{fi}(d) : \forall f, i \\ MC_{fih}(y_{yih}) : \forall f, i, h \\ \rho.(z) \otimes e : \end{bmatrix}$$

In the last line of the above definition, $\rho.$ is the vector of all $\rho_a, a \in A,$ $\otimes$ denotes the Kronecker product, and $e$ is the vector of ones in $R^M.$ This produces $M$ copies of $\rho.$

Now put

$$K \equiv \left( \prod_{f=1}^{M} \tilde{K}_f \right) \cap \Omega$$
From the formulation above we have a pair \((K,F)\) with which to define a VI problem.

**Proposition.** Assume that each function \(\theta_f(x)\) is concave in \(x^f\) for each fixed but arbitrary \(\tilde{x}^f\). If \(x \equiv (d, y, z)\) solves the VI \((K,F)\), then \(x\) is an equilibrium point of the oligopolistic electricity model.

**Proof.** Let \(x\) solve the VI \((K,F)\). We need to verify that for each \(f\), the subvector \(x^f\) solves the firm’s maximization problem with \(x^g\) fixed for \(g \neq f\) and for \(\rho \equiv \rho(z)\). Let \(\hat{x}^f \in K_f(\tilde{x})^f\) be arbitrary. The vector \(\hat{x} \equiv (\hat{x}^f, \tilde{x})\) belongs to \(K\). Hence

\[
(\hat{x} - x)^T F(x) \geq 0,
\]

and this reduces to

\[
(\hat{x}^f - x^f)^T \nabla_{x^f} \theta_f(x) \leq 0.
\]

The claim now follows from the concavity of \(\theta_f(\cdot, \tilde{x}^f)\).
Variants of this model

1. The firms that generate the electricity do not control the means of transmission. Instead, they sell what they produce to an Independent System Operator (ISO) and pay a fee to the ISO for use of the transmission lines. The ISO distributes the electricity to the (demand) nodes in order to maximize its own profit, taking account of the transmission link capacities and the total amount of power to be distributed. The ISO becomes another player in a Nash game. (One needs to take account of market clearing conditions and the like.)

2. There could also be arbitragers in the market. The arbitragers will eliminate spatial price differences. The modeling of the problem with arbitragers may depend on how the firms deal with the arbitragers; it can lead to a VI formulation or to a Nash game in which each player solves a nonconvex constrained optimization problem called an MPEC.