Stanford University, Management Science and Engineering (and ICME)

# MS&E 318 (CME 338) Large-Scale Numerical Optimization

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Homework 2, Due Monday April 23

`http://stanford.edu/class/msande318/homework.html`

Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and $b \in \mathbb{R}^n$ ($b \neq 0$). Let the $k$th Krylov subspace be $\mathcal{K}_k \equiv \text{span}\{b, Ab, A^2b, \dots, A^{k-1}b\}$, and let $(T_k, H_k, V_k)$ denote the matrices obtained by applying the Lanczos process to $(A, b)$ for $k$ steps, as defined in the notes. Assume that we use exact arithmetic and that the process terminates with $\beta_{k+1} = 0$ for some $k = \ell \le n$.

## Exercise 1

If $A$ has $m < n$ distinct eigenvalues, show that the Lanczos process for $(A, b)$ terminates with $\ell \le m$. (If you wish, you may prove that the dimension of the Krylov subspaces $\mathcal{K}_k(A, b)$ cannot exceed $m$.)

## Exercise 2

Show that if $A$ is changed to $A - \sigma I$ for some scalar shift $\sigma$, $T_k$ becomes $T_k - \sigma I$ and $V_k$ is unaltered.

## Exercise 3

Using $AV_k = V_{k+1}H_k$ for $k < \ell$ and the definitions of $\{\alpha_k, \beta_k, v_k\}$, show that the columns of $V_k$ are orthonormal ($k \le \ell$).

## Exercise 4

Show that the following subproblems are equivalent for defining how CG chooses $x_k$ when $A$ is positive definite:

$$\begin{array}{ll} (1) & \text{minimize } \tfrac{1}{2}x_k^TAx_k - b^Tx_k \text{ such that } x_k \in \mathcal{K}_k \\ (2) & \text{minimize } \|r_k\|_{A^{-1}} \text{ such that } x_k \in \mathcal{K}_k \\ (3) & \text{find } x_k \text{ such that } x_k \in \mathcal{K}_k \text{ and } r_k \perp \mathcal{K}_k, \end{array}$$

where $r_k = b - Ax_k$ and $\|w\|_{A^{-1}} = \sqrt{w^TA^{-1}w}$ for all $w \in \mathbb{R}^n$.

## Exercise 5 (extra credit)

Prove that $T_\ell$ is nonsingular if and only if $b \in \text{range}(A)$.
Deduce that $\beta_{\ell+1} = 0$ is a "lucky breakdown" for CG, MINRES, and SYMMLQ if $b \in \text{range}(A)$.