

**MS&E 319: Matching Theory**  
**Homework 2**  
**Due: May 20, 2019**

**Problem 1.** Derive the Hall's marriage theorem from Tutte's theorem.

**Problem 2.** Prove that if a simple graph  $G$  on an even number of points  $p$  has more than  $\binom{p-1}{2}$  edges, then it has a perfect matching.

**Problem 3.** Consider a weighted complete bipartite graph with the same number of nodes on each side. To find the minimum weight perfect matching, the greedy algorithm (which repeatedly finds the minimum weight edge disjoint from all the previously selected edges) can lead to a solution whose weight divided by the optimum weight can be arbitrarily large (even for graphs with 2 vertices on each side of the bipartition).

Suppose now that the weight of each edge comes from a metric, even just a line metric. More precisely, suppose that the bipartition is  $A \cup B$  with  $|A| = |B| = n$  and the  $i$ th vertex of  $A$  (resp. the  $j$ th vertex of  $B$ ) is associated with  $a_i \in \mathbb{R}$  (resp.  $b_j \in \mathbb{R}$ ). Suppose that the weight of the edge between these vertices is given by  $w_{i,j} = |a_i - b_j|$ .

Consider the greedy algorithm: select the closest pair of vertices, one from  $A$  and one from  $B$ , match them together, delete them, and repeat until all vertices are matched. For these line metric instances, is the weight of the greedy solution always upper bounded by a constant (independent of  $n$ ) times the weight of the optimum assignment? If so, prove it; if not, give a family of examples (parameterized by  $n$ ) such that the corresponding ratio become arbitrarily large.

**Problem 4. (extra credit)** Let  $L_n$  be the number of latin squares of size  $n$ . Prove that

$$\prod_{k=1}^n (k!)^{n/k} \geq L_n \geq \frac{(n!)^{2n}}{n^{n^2}}.$$