1) a) Because supply and demand are smooth, the supply curve for one competitive firm is determined by equality between marginal production costs and price. Hence, 
\[
\frac{dc}{dp} = y = p \Rightarrow y = p.
\]
b) Let \(y_c\) be the total output from the competitive sector. Since the firms are identical, we know that \(y_c = 50y\). Hence, 
\[
\frac{y_c}{p} = p \Rightarrow y_c = 50p.
\]
c) At a price \(p\), demand will be \(1000 - 50p\). Production by the competitive sector will be \(50p\). Let \(y_m\) be the monopoly quantity. Assert that the monopolist will supply all demand in excess of the competitive supply, or \(y_m = D - y_c\). Hence, 
\[
y_m = 1000 - 50p - 50p \Rightarrow y_m = 1000 - 100p.
\]
d & e) The monopolist attempts to maximize profit, \(\pi_m\).
\[
\pi_m =\]
\[
r_m - c_m = p\ y_m(p) - c_m(y_m(p)) =
\]
\[
p(1000 - 100p) - c_m(y_m(p)) =
\]
\[
1000p - 100p^2 - c_m(y_m(p))
\]
Because demand and supply are smooth, we can take the first-order conditions, equating marginal costs with marginal revenue, to find the global optimum.
\[
\frac{\partial \pi}{\partial p} = 1000 - 200p = 0 \Rightarrow p = 5 \Rightarrow y_m = 500.
\]
f) The competitive sector will provide 
\[
y_c = 50p = 250.
\]
g) Total output will be 
\[
y = y_c + y_m = 250 + 500 = 750.
\]
2) a) In a competitive equilibrium, it will be the case that \(p = MC\). Thus, 
\[
100 - Y = 0 \Leftrightarrow Y = 100.
\]
b) If each firm behaves as a Cournot competitor, it will solve: 
\[
\max_{y_i} (100 - y_i - y_{-i})y_i
\]
holding \(y_{-i}\) constant. For firm 1 this yields FOC: 
\[
100 - 2y_1 - y_2 = 0
\]
\[
\Rightarrow y_1^*(y_2) = \frac{100-y_2}{2}
\]
c) To find the Cournot equilibrium output, we solve both firms FOCs simultaneously: 
\[
100 - 2y_1 - y_2 = 0
\]
\[
100 - y_1 - 2y_2 = 0
\]
\[
\Rightarrow y_1^c = y_2^c = \frac{100}{3}
\]
d) If the two work as a cartel, they are solving together the problem

$$\max_{y_1,y_2} (100 - y_1 - y_2)y_1 + (100 - y_1 - y_2)y_2$$

Notice that this is the same as the monopolists problem:

$$\max_y (100 - Y)Y$$

So we know that $$p^I \frac{\partial}{\partial y} + 1 = 0$$ where $$\frac{\partial}{\partial y} = -\frac{100-Y}{Y}$$. This equation is solved when $$Y = 50$$. So, the cartel output is $$y_1 + y_2 = Y^m = 50$$.

e) We begin with firm 1. Suppose firm 2 has already chosen to produce amount $$y_2$$. Then firm 1 will produce $$y_1^*(y_2)$$ as derived in part b) because $$y_1^*(y_2)$$ is the optimal amount for firm 1 to produce given that firm 2 is producing $$y_2$$.

Firm 2 is able to anticipate what firm 1 will do, and takes this into consideration when choosing $$y_2$$. Firm 2 solves.

$$\max_{y_2} y_2 \left(100 - y_1^*(y_2) - y_2\right)$$

$$\Rightarrow \text{FOC: } 100 - \frac{100-y_2}{2} - y_2 + \left(\frac{1}{2} - 1\right)y_2 = 0$$

$$\Rightarrow y_2^5 = 50, y_1^1 = y_1^*(y_2^5) = 25$$

3) The British firm is a profit maximizer operating with a fixed price, $$p^*$$: the market is competitive. The British firm solves the maximization

$$\max_y \pi = p^* y - c(w, r, y)$$

a) With an import tax and export subsidy, the maximization is now,

$$\max_y \pi = (p^* + s - t)y - c(w, r, y)$$

The optimal $$y$$ will remain unchanged from the no-intervention case if $$t = s$$.

b) With the capital subsidy, the optimization becomes

$$\max_y \pi = (p^* - t)y - c(w, r - s, y)$$

Take the first-order conditions:

$$\frac{\partial \pi}{\partial y} = p^* - t - \frac{\partial c(w, r - s, y)}{\partial y} = 0 \Rightarrow p^* - t = \frac{\partial c(w, r - s, y)}{\partial y}$$

c) Now we are going to look at the change in $$t$$ with changes in the subsidy level. Take the derivative of the equality in (b) with respect to $$s$$.

$$\frac{\partial \pi}{\partial s} = \frac{\partial^2 c(w, r - s, y)}{\partial y \partial s} = \frac{\partial c(w, r - s, y)}{\partial y} \frac{\partial (r - s)}{\partial s} = \frac{\partial K(w, r - s, y)}{\partial y}$$

Note that we used Shepherd’s lemma in the final step.

d) With constant returns to scale, $$K(w, r - s, y) = K(w, r - s, 1) y$$. Hence,

$$\frac{\partial \pi}{\partial s} = K(w, r - s, 1)$$.

e) If the factor of production is inferior, then $$\frac{\partial K}{\partial y} < 0$$. Hence, the higher is the British capital subsidy, the lower need be the American import tax.

4) a & b) First, look at the equality between the unconstrained monopoly and licensing solutions. Say that $$p_m$$ is the monopoly price associated with marginal cost $$m$$. The monopolist can arrive at the solution by optimizing over price or over quantity. Optimizing over price gives the first-order conditions
If the firm licenses, the market price is given by \( m + L \), assuming that \( m + L < M \). The firm’s profit per unit is given by \( L \). Total profit is given by

\[
(2) \quad L q(m + L)
\]

The optimal solution is given by the first-order conditions,

\[
(3) \quad q(m + L) + L q'(m + L) = 0.
\]

Substitution \( p = m + L \) gives

\[
(4) \quad q(p) + (p - m)q'(p) = 0,
\]

which is identical to (1), the monopolist first-order conditions. Hence, the unconstrained optimal solution with licensing is identical to that without licensing, or \( p_m = m + L^* \).

The problem splits into two possible scenarios. Either \( p_m > M \) or \( p_m < M \). Let \( p_c \) be the competitive price associated with the pre-innovation marginal cost, \( M \).

Case 1: \( p_m > M \) (constrained)

Without licensing, the optimal solution is given by \( p = M - \epsilon \), where \( \epsilon \) is small. The innovator can’t price above \( M \) because the competitors will undercut the innovator. Since the unconstrained monopoly price is greater than \( M \) and the profit function is concave, the innovator wants to price as close to \( M \) as possible. A price below \( M \) will have lower profits than \( p = M \) (or \( \epsilon \) below \( M \), to be precise). Hence the optimal profit is given by

\[
(5) \quad (M - \epsilon) q(M - \epsilon) - m q(M - \epsilon) = (M - \epsilon - m) q(M - \epsilon)
\]

The profit function with licensing is also concave with the optimal unconstrained license price lying above \( M - m \). Hence, the constrained optimal license price is given by \( L^* = M - m \). Profit by licensing is given by

\[
(6) \quad L^* q(m + L^*) = (M - m) q(M)
\]

Hence, profits are at least as high with licensing as when the innovator produces his or herself.

Case 2: \( p_m < M \) (unconstrained)

First show that if \( m \) is small enough \( p_m < M \). Note that a monopolist maximizes \( p(q)q - m q \), using quantity as the variable of choice now. If marginal cost is \( m \), the first-order conditions give \( p'(q_m)q_m - p(q_m) = m \). We know that \( p'(q_c)q_c - p(q_c) > 0 \), where \( q_c \) is the competitive quantity with marginal cost \( M \). Set \( m = 0 \), and \( p'(q_c)q_c - p(q_c) > 0 = m \), which implies that \( p_m < p_c \).

(1) and (4) show that the optimal solution is the same with and without licensing in the unconstrained case.

5) The following results were generated using Mathematica©.

a) In this case Acme faces the demand function:

\[
\text{In[1]} := \text{w[p]} := 6 (p^{-1.5} + p^{-3} + p^{-9} + p^{-36})
\]

which is \( \sum_{i=1}^{4} w_i(p) \). The firm will choose \( p \) in order maximize profit. (Notice that we make use of the duality between price and quantity choices to simplify the mathematics. If we tried to model the monopolist as making a quantity choice, we would have a very difficult time characterizing demand in closed form!)

\[
\text{In[2]} := \text{S} := \text{p w[p]} - \text{w[p]}; \text{Solutions} = \text{Solve[D[S, p] == 0, p]}
\]
Thus, for each type we know that
\[ p_{i} \]
Out\[2\]=  
\[ \{p \rightarrow 1.14983 - 0.528164 \, i, \} \{p \rightarrow 1.14983 + 0.528164 \, i, \} \],
\[ \{p \rightarrow 1.02889 - 0.228465 \, i, \} \{p \rightarrow 1.02889 + 0.228465 \, i, \} \],
\[ \{p \rightarrow 0.957198 - 0.483327 \, i, \} \{p \rightarrow 0.957198 + 0.483327 \, i, \} \],
\[ \{p \rightarrow 0.800202 - 0.686583 \, i, \} \{p \rightarrow 0.800202 + 0.686583 \, i, \} \],
\[ \{p \rightarrow 0.779298 - 1.88376 \, i, \} \{p \rightarrow 0.779298 + 1.88376 \, i, \} \],
\[ \{p \rightarrow 0.62158 - 0.842573 \, i, \} \{p \rightarrow 0.62158 + 0.842573 \, i, \} \],
\[ \{p \rightarrow 0.417681 - 0.960939 \, i, \} \{p \rightarrow 0.417681 + 0.960939 \, i, \} \],
\[ \{p \rightarrow 0.190721 - 1.03725 \, i, \} \{p \rightarrow 0.190721 + 1.03725 \, i, \} \],
\[ \{p \rightarrow 0.0604266 - 1.06544 \, i, \} \{p \rightarrow 0.0604266 + 1.06544 \, i, \} \],
\[ \{p \rightarrow 0.185136 - 1.31966 \, i, \} \{p \rightarrow 0.185136 + 1.31966 \, i, \} \],
\[ \{p \rightarrow 0.318456 - 1.00912 \, i, \} \{p \rightarrow 0.318456 + 1.00912 \, i, \} \],
\[ \{p \rightarrow 0.531064 - 0.898902 \, i, \} \{p \rightarrow 0.531064 + 0.898902 \, i, \} \],
\[ \{p \rightarrow 0.712857 - 0.758741 \, i, \} \{p \rightarrow 0.712857 + 0.758741 \, i, \} \],
\[ \{p \rightarrow 0.869082 - 0.591951 \, i, \} \{p \rightarrow 0.869082 + 0.591951 \, i, \} \],
\[ \{p \rightarrow 0.995807 - 0.0714481 \, i, \} \{p \rightarrow 0.995807 + 0.0714481 \, i, \} \],
\[ \{p \rightarrow 1.011 - 0.423217 \, i, \} \{p \rightarrow 1.011 + 0.423217 \, i, \} \],
\[ \{p \rightarrow 1.03835 - 0.281723 \, i, \} \{p \rightarrow 1.03835 + 0.281723 \, i, \} \{p \rightarrow 2.13252 \} \}

There are many potential solutions to this problem, but only one of them is real. Thus, we take the uniform price to be:

\[ w_{\text{star}} = \text{solution}[[29]][[1]][[2]] \]

Out\[3\]=  
\[ 0.995807 - 0.0714481 \, i \]

The number of widgets sold at this price is:

\[ w[p_{\text{star}}] \]

Out\[4\]=  
\[ 2.55196 \]

b) In this case Acme charges each type a different price. The firm will solve a monopolist’s problem for each type of consumer. Thus, for each type we know that \[ p_{i} = \frac{p}{w_{i}} \]. From this equation we can solve for \[ p_{i} \] and plug result back into the demand to get \[ w_{i} \].

\[ w2[p_{\_}] := 6p^{2}\]

Out\[6\]=  
\[ 1.5 \]

Out\[6\]=  
\[ 1.77778 \]

\[ w3[p_{\_}] := 6p^{3}\]

Out\[7\]=  
\[ 1.125 \]

Out\[7\]=  
\[ 2.07864 \]
To see who does better and who does worse is straightforward; we simply compute the welfare for each party in each case and compare them. For Acme LLC, we compare profits:

\[\text{In[9]} := N[p_{\text{star}} w_{\text{star}}] - p_{\text{star}}\]

\[N[p_{\text{star}} w_{\text{star}}] - w_{\text{star}} p_{\text{star}} + p_{\text{star}} w_{\text{star}} - p_{\text{star}} w_{\text{star}} + p_{\text{star}} w_{\text{star}} - w_{\text{star}} p_{\text{star}}\]

\[\text{Out[9]} = 3.30399\]

\[\text{Out[9]} = 9.071\]

As we expected, Acme does significantly better when price discriminating.

For each of the consumer types, we compare the consumer surplus in each case. But we have to be careful—consumer surplus is most naturally found by integrating the inverse demand function \(p_i(w_i)\). Thus the consumer surplus in each case is \(\int_0^{\bar{w}} p_i(w) dw - p_i(\bar{w}) \bar{w}\) for appropriate value of \(\bar{w}\).

\[\text{In[10]} := p_{\text{star}} := \left(\frac{w}{6}\right)^{\frac{1}{3}}\]

\[\int p_{\text{star}}(w) dw - p_{\text{star}} w_{\text{star}} - \int p_{\text{star}}(w) dw - p_{\text{star}} w_{\text{star}}\]

\[\text{Out[10]} = 8.21741\]

\[\text{Out[10]} = 6.9282\]

So type 1 consumers are worse off when Acme price discriminates.

\[\text{In[11]} := p_{\text{star}} := \left(\frac{w}{6}\right)^{\frac{1}{3}}\]

\[\int p_{\text{star}}(w) dw - p_{\text{star}} w_{\text{star}} - \int p_{\text{star}}(w) dw - p_{\text{star}} w_{\text{star}}\]

\[\text{Out[11]} = 0.659685\]

\[\text{Out[11]} = 1.33333\]

\[\text{In[12]} := p_{\text{star}} := \left(\frac{w}{6}\right)^{\frac{1}{3}}\]

\[\int p_{\text{star}}(w) dw - p_{\text{star}} w_{\text{star}} - \int p_{\text{star}}(w) dw - p_{\text{star}} w_{\text{star}}\]

\[\text{Out[12]} = 0.00175357\]

\[\text{Out[12]} = 0.292308\]
In[13]:= \[\text{p4} \left[ \frac{w}{6} \right] \]

\text{Integrate[p4[w], \{w, 0, w4[pstar]\}] - pstar w4[pstar]}

\text{Integrate[p4[w], \{w, 0, w4star\}] - p4star w4star}

Out[13] = 5.28263 \times 10^{-13}

Out[13] = 0.0639554

But type 2, 3, and 4 consumers are better off when Acme price discriminates.

6) a) In a perfectly competitive market \( p = MC \), so

\[ A - B q = m \]

\[ \Rightarrow q^{pc} = \frac{A-m}{B} \text{ and } p^{pc} = m. \]

b) In a cournot equilibrium, each producer \( i \) solves

\[ \max q_i (A - B Q)q_i - m q_i \]

where \( Q = \sum_{i=1}^{n} q_i \) and holding \( q_j \) constant \( \forall j \neq i \). This yields the \( n \) FOCs:

\[ A - B Q - m = B q_i, \forall i \]

\[ \Rightarrow q_i = q_j \equiv q_o, \forall i, j \]

substituting we get

\[ A - B n q_o - m = B q_o \]

\[ \Rightarrow q_o = \frac{A-m}{(n+1)B} \]

\[ \Rightarrow Q = \frac{n(A-m)}{(n+1)B} \]

\[ \Rightarrow p = A - \frac{n(A-m)}{(n+1)} \]

c) In the limit these values are:

\[ \lim_{n \to \infty} Q = \frac{(A-m)}{p} = Q^{pc} \]

\[ \lim_{n \to \infty} p = A - (A - m) = m = p^{pc} \]