1) a) i) \[ \text{MRS}_A = \text{MRS}_B \]

\[ \Rightarrow \frac{\partial U_A}{\partial x_1} = \frac{\partial U_B}{\partial x_2} \Rightarrow \frac{x_1}{x_2} = \frac{(5-x_2^A)}{(5-x_2^B)} \]

\[ \Rightarrow x_2^A(x_1^A) = \frac{5x_1^A}{2x_1^A-5} \text{ (with } x_1^B = 5 - x_1^A \text{ and } x_2^B = 5 - x_2^A) \]

ii) Because the functions are Cobb-Douglas, we know that:

\[ x_1^A(p_1, p_2) = \frac{p_1 + 2p_2}{2p_1}, \quad x_2^A(p_1, p_2) = \frac{p_1 + 2p_2}{2p_2}, \]

\[ x_1^B(p_1, p_2) = \frac{4p_1 + 3p_2}{4p_1}, \quad x_2^B(p_1, p_2) = \frac{12p_1 + 9p_2}{4p_2} \]

\[ \Rightarrow \left\{ \begin{array}{c}
\frac{p_1 + 2p_2}{2p_1} + \frac{4p_1 + 3p_2}{4p_2} = 5 \\
\frac{2p_1}{2p_2} + \frac{12p_1 + 9p_2}{4p_2} = 5 \\
\end{array} \right. \]

\[ \Rightarrow \left\{ \begin{array}{c}
p_1 = \frac{1}{2} \\
p_2 = \frac{5}{4} \end{array} \right. \\
\Rightarrow \left\{ \begin{array}{c}x_1^A = \frac{5}{2}, \quad x_2^A = \frac{5}{4}, \quad x_1^B = \frac{10}{4}, \quad x_2^B = \frac{15}{4} \end{array} \right. \]

iii)

b) i) Because consumer A has a Leontief utility function, we know that an allocation is PO if and only if \( x_2^A = x_1^B \) (with \( x_1^B = 5 - x_1^A \) and \( x_2^B = 5 - x_2^A \)).
Proof: Suppose \( x_A^1 > x_A^2 \). \( U_A(x_A^1, x_A^2) = U_A(x_A^1, x_A^1 - (x_A^2 - x_A^1)) \), so we can give \( B (x_A^2 - x_A^1) \) of good 2, making \( B \) better off without making worse off. A symmetric argument holds if \( x_A^2 < x_A^1 \). Thus, an allocation is PO only if \( x_A^2 = x_A^1 \). Any reallocation that reduces the amount of either good consumed by \( A \) away give \( A \) lower utility (even if that allocation increases the amount of the other good consumed by \( A \)). Thus, it is not possible to make \( B \) better off without making \( A \) worse off. The only way to make \( A \) better off is to give her more of each good. But this reallocation give less of each good to \( B \), which makes \( B \) worse off. Thus, any allocation that has \( x_A^2 = x_A^1 \). QED

ii) Because the functions are Leontief and Cobb-Douglas, respectively, we know that:

\[
\begin{align*}
x_1^A(p_1, p_2) &= x_2^A(p_1, p_2) = \frac{p_1 + 2p_2}{p_1 + p_2}, \\
x_1^B(p_1, p_2) &= \frac{4p_1 + 3p_2}{4p_1}, \\
\text{and } x_2^B(p_1, p_2) &= \frac{12p_1 + 9p_2}{4p_2},
\end{align*}
\]

\[ \Rightarrow \left\{ \begin{array}{l} \frac{p_1 + 2p_2}{p_1 + p_2} + \frac{4p_1 + 3p_2}{4p_1} = 5 \\
\frac{p_1 + 2p_2}{p_1 + p_2} + \frac{12p_1 + 9p_2}{4p_2} = 5 \\
\frac{p_1}{p_2} = \frac{1}{3}
\end{array} \right. \]

\[ \Rightarrow \left\{
\begin{array}{l}
x_1^A = \frac{7}{4}, x_2^A = \frac{7}{4}, x_1^B = \frac{13}{4}, x_2^B = \frac{13}{4}
\end{array} \right. \]

iii) P. O. points

Budget

\( \begin{array}{c} x_A^1 \quad x_A^2 \quad x_B^1 \quad x_B^2 \\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \end{array} \)

\( U_A \)

\( U_B \)

\( \Rightarrow \left\{ \frac{\frac{x_B}{x_A}}{x_A} \right\} \)

c) i) We are looking for points of tangency, however, the marginal rate of substitution for \( B \) \( \frac{\frac{x_B}{x_A}}{x_A} \)
does not equal 0 or oo (unless the utility for B equals 0, which is a degenerate case). Thus, we know that point of tangency will be at an extreme point, where the utility curve of B intercepts either the $x_1^A$ axis or the $x_2^A$ axis. The point of tangency will be at the larger of these two extremes.

\[
x_1^A \text{ intercept } = 5 - \frac{y}{5}
\]

\[
x_2^A \text{ intercept } = 5 - \frac{y}{5}
\]

\[
5 - \frac{y}{5} \geq 5 - \frac{y}{5} \text{ as long as } y \leq 5
\]

So, the point of tangency can occur anywhere along the $x_1^A$ axis. That is, the PO allocations are $(x_1^A, 0)$ for $x_1^A \in [0, 5]$ (with $x_1^B = 5 - x_1^A$ and $x_2^B = 5 - x_2^A$).

**Proof:** \(\Rightarrow\) Suppose the allocation is $x_1^A = k$, $x_2^A = 0$, $x_1^B = 5 - k$, $x_2^B = 5$. The only way to make A better off is to give her more of good 1, but this will make B strictly worse off because we have to take good 1 from him to give it to A but there is no more of good 2 to give him to compensate. The only way to make B better off is to give him more of good 1 (because there is not more of good 2 to give him). But this will make A worse off because we will have to take some good 1 away from her. \(\Leftarrow\) Suppose the allocation is $x_1^A = k$, $x_2^A = l$, $x_1^B = 5 - k$, $x_2^B = 5 - l, k < l$. This can’t be PO because we could give all 5 units of good 2 to B without reducing A’s utility. Symmetrically if $k > l$. Suppose the allocation is $x_1^A = 0$, $x_2^A = k$, $x_1^B = 5$, $x_2^B = 5 - k$. We could keep B’s utility constant and increase A’s utility by switching to the other extreme point as shown above. **QED**

**ii)** In this case, there may be multiple CEs. A PO allocation will be supportable as CE allocation if the line between the initial endowment allocationa and the PO allocation has slope greater than the MRS for B as the PO allocation and less than 1. That is:

\[
\frac{\bar{x}_1^B}{\bar{x}_2^B} \leq \frac{2x_1^A}{x_1^A-1} \leq 1
\]

\[
\Rightarrow 3 \leq x_1^A \leq \frac{35}{11}
\]

**iii)**

\[
\frac{\bar{x}_1^B}{\bar{x}_2^B} \leq \frac{2x_1^A}{x_1^A-1} \leq 1
\]

\[
\Rightarrow 3 \leq x_1^A \leq \frac{35}{11}
\]
d) i) Since these lines have different slopes, we know that a PO allocation will be an extreme point. Furthermore, since the slope of A preferences is steeper, we know this extreme point will lie on either the \( x_1 \) axis or the \( y_2 \) axis. Thus, the PO allocations are: \( (x_1^A \in [0, 5], x_2^A = 0) \) and \( (x_1^A = 5, x_2^A \in [0, 5]) \) (with \( x_1^B = 5 - x_1^A \) and \( x_2^B = 5 - x_1^A \)).

**Proof:** Suppose the allocation is \( x_1^A = k, x_2^A = l \) with \( x_1^B = 5 - x_1^A \) and \( x_2^B = 5 - x_1^A \) and with \( 0 < k < 5 \) and \( l > 0 \). It is possible to give A more of good 1 and \( \frac{L}{3} \) less of good 2 which strictly improves A’s welfare and leaves B indifferent. \( \Rightarrow \) Suppose the allocation is one of the first type of allegedly PO allocations. The only way to improve A’s welfare is to give her more of good 1 or to give her \( \frac{L}{3} \) more of good 2 and to take less than \( \Delta \) of good 1 away. But either of these reallocations would make B strictly worse off. Conversely, the only way to improve B’s welfare is to give him more \( \frac{L}{3} \) of good 1 and take less \( \frac{L}{3} \) of good 2 away. But this will make A strictly worse off. Now suppose the allocation is one of the first second type of allegedly PO allocations. Improvement for A can only come if we give her \( \Delta \) more of good 2 and take away less than \( \Delta \) of good 1. But this will make B strictly worse off. Conversely, we can only make B better off by giving him more of good 2 or giving him \( \frac{L}{3} \) more of good 1 and taking away less than \( \frac{L}{3} \) of good 2. But either of these would make A strictly worse off. **QED**

ii) Take any PO allocation and a given endowment point. For the allocation to be supportable in a competitive equilibrium, it must be possible to draw a line between the endowment point and the allocation whose slope is between \( \frac{1}{3} \) and 1. To restate this, then \( \frac{1}{3} \leq \frac{x_1^A}{x_2^A} \leq 1 \). So, all of the PO allocations \{ \( (x_1^A \in [3, 5], x_2^A = 0) \) and \( (x_1^A = 5, x_2^A \in [0, \frac{5}{3}]) \) \} are supportable as competitive equilibrium points for the initial endowments.

iii)
2) At a C.E. it must be the case that \( \frac{p_1}{p_2} = \text{MRS}_{3,1} \). That is,
\[
\frac{p_1}{p_2} = \frac{5}{5} = \frac{1}{2}
\]
3) I’ll do this problem as an extension of the exchange theory that we have been working on. There are two goods in the economy, good 0 and good 1. Good zero can be turned into good 1 on a one-for-one bases; you can put good 0 under your mattress. Alternatively, individuals can trade goods. Individual A may give some of good 0 to individual B in exchange for some good 1. The price of the exchange defines the interest rate: \( p_0 = (1 + r)p_1 \). If \( r \) is positive, then good 0 costs more than good 1 and vice versa.

a. The two individuals face similar maximization problems. For individual A, we have,
\[
\max u_A = \ln c_A^0 + b \ln c_A^1
\]
subject to \( p_0 c_A^0 + p_1 c_A^1 = p_0 w_0 \)

For individual B we have
\[
\max u_B = \ln c_B^0 + b \ln c_B^1
\]
subject to \( p_0 c_B^0 + p_1 c_B^1 = p_1 w_1 \)

b. To solve the problem first form the lagrangians and then obtain the demand functions. The lagrangians are given by,
\[
L^A = \ln c_A^0 + b \ln c_A^1 - \lambda (p_0 c_A^0 + p_1 c_A^1 - p_0 w_0)
\]
\[
L^B = \ln c_B^0 + b \ln c_B^1 - \lambda (p_0 c_B^0 + p_1 c_B^1 - p_1 w_1)
\]
Taking the first order conditions for each individual - \( \frac{\partial L}{\partial c_0} = 0 \), \( \frac{\partial L}{\partial c_1} = 0 \), and \( \frac{\partial L}{\partial \lambda} = 0 \)– and solving gives the demand functions,
\[
c_A^0 = \frac{w_0}{1+b} \quad c_A^1 = \frac{p_0 b}{p_1} \frac{w_0}{1+b}
\]
Now there are three possibilities for the interest rate. First, it can not be negative. If it is negative, then individual A will store unused good 0 under his mattress for use in period 1. Individual B will demand some of good 0, but none will be available since it will be under the mattress. Supply won’t equal demand and there is no competitive equilibrium.

Second it may be that the interest rate is positive. In this case, individual A will lend all unused good 0 rather than storing it under the mattress. Competitive equilibrium exists when the amount of good 0 that individual A sells is equal to the amount that individual B purchases, or \( c^A_0 + c^B_0 = w_0 \). Substituting the demand functions into this constraint gives, \( p_0 = p_1 \frac{w_1}{b w_0} \), or \( r = \frac{w_1}{b w_0} - 1 \). This is the interest rate if \( \frac{w_1}{b w_0} \geq 1 \). If \( \frac{w_1}{b w_0} < 1 \), \( r < 0 \) and the positive interest rate assumption does not hold.

So what if \( \frac{w_1}{b w_0} < 1 \)? Let us assume a third case: the interest rate is zero, or \( p_0 = p_1 \). In this case, individual A will be perfectly happy either to store good 0 under his mattress or to sell some. Both ways will give him the same amount of good 1. Hence, supply need not equal demand for good 0, or \( c^A_0 + c^B_0 \leq w_0 \). To verify, we plug in the demand equations to get, \( c^A_0 + c^B_0 = w_1 + b + w_1 1 + b \). Remember that \( w_1 < b w_0 \) in this case so that \( c^A_0 + c^B_0 \leq w_0 \). Hence, individual A does put some money under his mattress. To be sure that second-period demand matches supply, we have, \( c^A_1 + c^B_1 = w_1 + (w_0 - c^A_0 + c^B_0) \)

Substituting gives,

\[
\frac{b (w_0 + w_1)}{1 + b} = \frac{b (w_0 + w_1)}{1 + b}.
\]

Hence, we have demand equal to supply and the market is in equilibrium.

Note: Here’s a more general way to way to set up the problem. Assume a two-good economy, as I have done, but note that individual A - or, in fact, both individuals - has a production technology that will take one unit of good 0 and turn it into one unit of good 1. He must decide whether to produce good 1 using good 0 or whether to purchase good 1 on the market by selling good 0.

4) For solution, see Varian, page A32, problem 18.1.

5) a) Economic incentives will lead to using all the resources available. So this is just like problem #1 with firms substituted for consumers and technology functions substituted for utility functions. That is, at a C. E. we expect that the ratio of prices will now be equal to the technical rate of substitution for each firm. That is, we expect the technical rates of substitution to be equal. Because no initial endowment has been specified, we know from the second theorem of welfare economics that any such point can be supported by a C. E. For simplicity suppose that firm A produces Food.

\[
\frac{P_N}{P_L} = \frac{s \sqrt{FP}}{s \sqrt{LP}} = \frac{-16 (1000-L_P)^{3/2}}{-4(3000-L_P)^{3/2}}
\]

So the points that may be supported in equilibrium may be characterized by the relationship:

\[
N_F(L_P) = \frac{1000 L_P}{3 (4000-L_P)}
\]
b) From part a) we have \( \frac{p_N}{p_L} = \frac{L_C}{N_F} = 4 \frac{L_C}{N_F} \).

c) From part a) \( N_F(L_F) = \frac{1000 L_F}{3(4000-L_F)} \). From part b) \( \frac{p_N}{p_L} = \frac{L_F}{N_F} = \frac{L_F}{1000 L_F} \).

\[ L_F = 4000 - \frac{1000}{3} \frac{p_N}{p_L} \]

\[ N_F = \frac{1000 L_F}{3(4000-L_F)} = 4000 \left( \frac{p_N}{p_L} \right)^{-1} - \frac{1000}{3} \]

\[ L_C = 3000 - L_F = \frac{1000}{3} \frac{p_N}{p_L} - 1000 \]

\[ N_C = 1000 - N_F = \frac{4000}{3} - 4000 \left( \frac{p_N}{p_L} \right)^{-1} \]

d) From part c) we can see that food production will be zero iff \( \frac{p_N}{p_L} \geq 12 \), and conversely clothing production will be zero iff \( \frac{p_N}{p_L} \leq 3 \).

e) Since the production functions are Cobb-Douglas, we know that they generate cost functions of the form:

\[ c_F(P_N, P_L, F) = 2 \sqrt{P_N P_L} F \]

\[ c_C(P_N, P_L, C) = 2 \left( \frac{1}{4} \right)^{0.8} + \left( \frac{1}{4} \right)^{-0.2} \right] P_L^{0.2} P_F^{0.8} \]

In a competitive equilibrium we expect that \( P_F = MC_F \) and \( P_C = MC_C \). So:

\[ P_F(P_N, P_L) = 2 \sqrt{P_N P_L} \]

\[ P_C(P_N, P_L) = 2 \left( \frac{1}{4} \right)^{0.8} + \left( \frac{1}{4} \right)^{-0.2} \right] P_L^{0.2} P_F^{0.8} \]

Furthermore:

\[ P_F(\Delta P_N, \Delta P_L) = 2 \sqrt{\Delta P_N \Delta P_L} = 2 \Delta \sqrt{P_N P_L} = \Delta P_F(P_N, P_L) \]

\[ P_C(\Delta P_N, \Delta P_L) = 2 \left( \frac{1}{4} \right)^{0.8} + \left( \frac{1}{4} \right)^{-0.2} \right] (\Delta P_L)^{0.2} (\Delta P_F)^{0.8} \]

\[ = 2 \Delta \left( \frac{1}{4} \right)^{0.8} + \left( \frac{1}{4} \right)^{-0.2} \right] P_L^{0.2} P_F^{0.8} \]

\[ = \Delta P_C(P_N, P_L) \]

So the output prices are homogeneous of degree 1 in the input prices.