The classical mean-variance model
(Markowitz, 1952, 1959)

$i$          asset, $i = 1, \ldots, n$
$\tilde{R}_i$ random rate of return, $r_i = E\tilde{R}_i$, $\sigma_{i,j} = \text{cov}(\tilde{R}_i, \tilde{R}_j)$
$x_i$        fraction of money invested in asset $i$

Convex quadratic problem:

$$\min \frac{1}{2} \sum_i \sum_j \sigma_{i,j} x_i x_j$$
$$\sum_i r_i x_i \geq \rho$$
$$\sum x_i = 1$$
$$x_i \geq 0$$

where $\rho$ is a desired minimum expected return
Large-scale portfolio optimization

Historically the Markowitz model was difficult to solve

- CAPM model, Sharpe (1970), single factor model
  \[ R_i \approx \alpha_i + \beta_i R_{\text{Market}} \]

- APT, Ross (1976), multi factor models
  \[ R_i \approx \alpha_i + \sum_k \beta_{ik} U_k \]

- Using historical observations of returns directly,
  MAD model, Konno et al. (1990),
  Markowitz et al. (1992)
Historical returns for measuring risk

\[ \bar{R} = [R_i^T] \quad [T \times n] \quad \text{matrix of historical returns} \]

\[ \bar{R} = \begin{pmatrix} \bar{R}_i \\ \bar{R}_i \\ \vdots \\ \bar{R}_i \end{pmatrix}, \quad \bar{R}_i = \frac{1}{T} \sum_{t=1}^{T} R_i^t \]

\[ R = \bar{R} - \bar{R} \]

\[ \tilde{Q}_H = \frac{1}{T} R^T R \quad [n \times n] \quad \text{estimated covariance matrix} \]

\[ \tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_n) \quad \text{expected return predictors} \]

In large-scale portfolio optimization, typically \( n \gg T \).
Markowitz model

Traditional formulation:

\[
\min \frac{1}{2} x^T \hat{Q}_H x \\
\quad e^T x = 1 \\
\quad \tilde{r} x \geq \rho, \quad l \leq x \leq u
\]

\(\hat{Q}_H\) estimated covariance matrix

\(l\) lower, \(u\) upper bound on holdings

Practical formulation:

\[
\min \frac{1}{2} \frac{1}{T} v^T v \\
\quad e^T x = 1 \\
\quad \tilde{r} x \geq \rho \\
\quad -R x + v = 0, \quad l \leq x \leq u
\]
Factor model

\[ R = UF + \epsilon \]

\( U = [T \times h] \) factors, \( F = [h \times n] \) factor loadings
\( \epsilon = [T \times n] \) residual returns

\[ \hat{Q}_H = \frac{1}{T} R^T R = \frac{1}{T} (F^T U^T UF + \epsilon^T \epsilon + F^T U^T \epsilon + \epsilon UF) \]

\[ \hat{Q}_{HF} = F^T \hat{Q}_F F + D, \quad D = \text{diag}(\sigma_i)^2 \]

\( \hat{Q}_{HF} \) estimated covariance matrix based on factor model,
\( \hat{Q}_F \) estimated factor covariance matrix
\( U \) and/or \( F \) may be obtained by using factor estimation (regression analysis) or principal component analysis
Factor model formulation in practice

\[ R_F = UF \quad \text{Returns explained by factors} \]

\[
\min \frac{1}{2} \left( x^T D x + \frac{1}{T} v^T v \right) \\
\quad \quad \quad e^T x = 1 \\
\quad \quad \quad \tilde{r} x \geq \rho \\
\quad \quad \quad -R_F x + v = 0, \quad l \leq x \leq u
\]
Factor model formulation in practice, alternative

\[\lambda \text{ Risk tolerance}\]

\[
\begin{align*}
\min -\bar{r}x + \frac{1}{2\lambda}(x^T Dx + \frac{1}{T}v^T v) \\
e^T x &= 1 \\
-R_F x + v &= 0, \quad l \leq x \leq u
\end{align*}
\]
Tracking error as a measure of risk

$w$ Benchmark portfolio, $e^T w = 1, \ w \geq 0$

$\lambda_A$ Tracking error tolerance

$$\begin{align*}
\min_{v} \ & -\tilde{r} x + \frac{1}{2 \lambda_A} \left\{ \frac{1}{T} v^T v + (x - w)^T D (x - w) \right\} \\
& e^T x = 1 \\
& -R_F (x - w) + v = 0, \quad l \leq x \leq u
\end{align*}$$
Mean absolute deviation (MAD) model  
(Konno et al., 1990)

\( i \) asset, \( i = 1, \ldots, n \)
\( \tilde{R}_i \) random rate of return, \( r_i = E \tilde{R}_i \)
\( x_i \) fraction of money investet in asset \( i \)

\[
\min \ E \left| \sum_i (\tilde{R}_i - r_i) x_i \right|
\]
\[
\sum_i r_i x_i \geq \rho
\]
\[
\sum x_i = 1
\]
\[
x_i \geq 0
\]

If asset returns are multivariate normals with mean \( r_i \), and covariance matrix \( Q = [\sigma_{i,j}] \), the MAD model has exactly the same solution as the Markowitz model.
MAD model with historical returns

Linear Program:

$$\begin{align*}
\min & \quad \frac{1}{T} e^T v \\
- R x & + v \geq 0 \\
+ R x & + v \geq 0 \\
\tilde{r} x & \geq \rho \\
e^T x & = 1, \quad 0 \leq x \leq u
\end{align*}$$

Solves faster, since LP.
Number of assets in portfolio can be controlled.
Expected utility optimization using factor models (Infanger, 2010)

\[
\begin{align*}
\max E & \ u(1 + (R^\omega_F + \epsilon)^T x) \\
Ax & = b, \ l \leq x \leq h
\end{align*}
\]

where

- \( R^\omega_F = F^T V^\omega \) vector of factor explained (part of the) returns
- \( F = [h \times n] \) factor loadings
- \( V^\omega \) vector of factors
- \( \epsilon \) vector of residual returns
- \( u \) concave, monotonically increasing utility function

- The problem can be solved efficiently.
- Allows for derivatives (e.g., options) in the portfolio.
Components of asset return

\[ R = R^\omega_F + \epsilon \]

\[ R^\omega_F = \alpha + \mu + z^\omega \]

where
- \( \alpha \) conditional means
- \( \mu \) unconditional means
- \( z^\omega \) demeaned factor explained variations
Calibrating the expected utility model to a benchmark

Quantify $\mu_B$ and calculate $z_B^\omega$ and $\sigma_B^2$ based on the benchmark weights.

Find $\mu$ such that the portfolio optimization model (for the above parameters) results in the benchmark portfolio. For $i = 1, \ldots, n$:

$$\mu_i = \mu_B - \frac{E \ u'_B (1 + \mu_B + z_B^\omega + \epsilon_B) (z_i^\omega + \epsilon_i - z_B^\omega - \epsilon_B)}{E \ u'_B (1 + \mu_B + z_B^\omega + \epsilon_B)}$$

• Implied means (Black and Litterman, 1992, Grinold, 1999)
Active portfolio management using expected utility maximization

Utility maximization portfolio optimization model:

$$\max E \ u(1 + \left(\frac{1}{\gamma_A} \alpha + \mu + \omega + \epsilon\right)^T x)$$

$$Ax = b, \ l \leq x \leq h$$

where

$\gamma_A$ active risk aversion parameter

• By adding the conditional mean forecast (based on the value of the factors), we take on active bets in a graduated fashion.
Active portfolio management using mean-variance optimization

\[
\max \left( \frac{1}{\gamma_A} \alpha + \mu \right)^T x - \frac{\gamma}{2} x^T Q x \\
Ax = b \quad l \leq x \leq h
\]

where \( \mu \) is calibrated as \( \mu = \gamma_B Q w \) and \( \gamma_B \) is the benchmark risk aversion \( (\gamma_B = 1/\lambda_B) \).
Commercial implementations

• Mean-variance (and MAD)
  – E.g., Barra, Northfield, Axioma
• Scenario-based
  – Allows also for options and other derivatives
  – E.g., Algorithmics (Ron Dembo, especially for risk evaluation)
• Issues of accuracy of parameter estimation
  – Richard Grinold: Mean-Variance and Scenario based Approaches to Portfolio Optimization, J. of Portfolio Management, Winter 1999
  – Robust portfolio optimization
Commercial implementations (cont.)
Commercial implementations (cont.)
Commercial implementations (cont.)
Commercial implementations (cont.)