LINEAR PROGRAMMING UNDER UNCERTAINTY

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Summary

The essential character of the general models under consideration is that activities are divided into two or more stages. The quantities of activities in the first stage are the only ones that are required to be determined; those in the second (or later) stages can not be determined in advance since they depend on the earlier stages and the random or uncertain demands which occur on or before the latter stage. It is important to note that the set of activities are assumed to be complete in the sense that, whatever be the choice of activities in the earlier stages (consistent with the restrictions applicable to their stage), there is a possible choice of activities in the latter stages. In other words it is not possible to get in a position where the programming problem admits of no solution.

The initial work on this paper was stimulated by discussions with A. Ferguson who proposed that linear programming methods be extended to include the case of uncertain demands for the problem of optimal allocation of a carrier fleet to airline routes to meet an anticipated demand distribution. The application of the theory found in this paper to his problem (discussed later under Example 4) will be the subject of a separate joint paper. The case of certain demands was discussed earlier [4].

A complete computation procedure is given for a special class of two-stage linear programming models in which allocations in the first stage are made to meet an uncertain but known distribution of demands occurring in the second stage. This case, applicable to many practical problems constitutes the principal part of the paper. Next, a class of models is considered where the activities are divided into two or more stages. The quantities of activities in the first stage are the only ones that can be determined in advance because those in the second and later stages depend on the outcome of random events. Theorems on convexity of the objective (cost) functions are established for the general m-stage case.

Example 1: Minimum Expected Cost Diet. A nutrition expert wishes to advise his followers on a minimum cost diet without prior knowledge of the prices pij. Since prices of food (except for general inflationary trends) are likely to show variability due to weather conditions, supply, etc., he wishes to assume a distribution of possible prices rather than a fixed price for each food, and determine a diet that meets specified nutritional requirements and minimizes expected costs.
Let $x_j$ be the quantity of $j^{th}$ food purchased in pounds, $p_j$ its price, and $a_{ij}$ be the quantity of the $i^{th}$ nutrient (e.g., vitamin A) contained in a unit quantity of the $j^{th}$ food, and $b_i$ the minimum quantity required by an individual for good health. Then the $x_j$ must be chosen so that

$$
\sum_{j=1}^n a_{ij}x_j \geq b_i, \quad x_j \geq 0 (i = 1, 2, \ldots, m)
$$

and the cost of the diet will be

$$
C = \sum_{j=1}^n p_jx_j.
$$

The $x_j$ are chosen before the prices are known so that the expected costs of such a diet are clearly

$$
\text{Exp } C = \sum_{j} \tilde{p}_jx_j
$$

where $\tilde{p}_j$ is its expected price. Since the $\tilde{p}_j$ are known in advance, the best choices of $x_j$ are those which satisfy (1), minimize (3). Hence in this case expected prices may be used in place of the distribution of prices and the usual linear programming problem solved.\footnote{In some applications, however, it may not be desirable to minimize the expected value of the costs if the decision has too great a variation in the actual total costs. H. Markowitz [5] in his analysis of investment portfolios develops a technique for computing for each possible expected value the minimum variance. This enables the investor to sacrifice some of his expectation to control his risks.}

Example 2: Shipping to an Outlet to Meet an Uncertain Demand.

Let us consider a simple two-stage case: A factory has 100 items on hand which may be shipped to an outlet at the cost of $\$1$ apiece to meet an uncertain demand $d_1$. In the event that the demand should exceed the supply, it is necessary to meet the unsatisfied demand by purchases on the local market at $\$2$ apiece. The equations that the system must satisfy are

$$
100 = x_{11} + x_{12}
$$

$$
d_2 = x_{11} + x_{21} - x_{22}
$$

$$
x_{ij} \geq 0
$$

where $x_{11}$ = number shipped from the factory, $x_{12}$ = number stored at factory; $x_{21}$ = number purchased on open market, $x_{22}$ = excess of supply over demand;

$$
d_2 = \text{unknown demand uniformly distributed between 70 and 80};
$$

$$
C = \text{total costs}.
$$

It is clear that whatever be the amount shipped and whatever be the demand $d_1$, it is possible to choose $x_{21}$ and $x_{22}$ consistent with the second equation. The unused stocks $x_{12} + x_{22}$ are assumed to have no value or are written off at some reduced value (like last year's model automobiles when the new production comes
To illustrate some of the concepts of this paper, a solution will be presented later.

**Example 3: A Three-Stage Case.**

For this purpose it is easy to construct an extension of the previous example by allowing the surpluses $x_{12}$ and $x_{22}$ to be carried over to a third stage, i.e.,

<table>
<thead>
<tr>
<th>Stage</th>
<th>Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st stage</td>
<td>100 = $x_{11} + x_{12}$</td>
</tr>
<tr>
<td>2nd stage</td>
<td>$d_2 = x_{11} - x_{12} + x_{22} + x_{24}$</td>
</tr>
<tr>
<td>3rd stage</td>
<td>$d_3 = x_{11} + x_{22} + x_{23} + x_{31} - x_{32}$</td>
</tr>
</tbody>
</table>

where $x_{32} =$ number shipped from factory in 2nd stage, $x_{24} =$ number stored at factory in 2nd stage;
70 = number produced 2nd stage;
$d_3 =$ unknown demand in 3rd stage uniformly distributed between 70 or 80;
x$_{31} =$ number purchased on the open market in 3rd stage, $x_{32} =$ excess of supply over demand in 3rd stage.$^2$

It will be noted that the distribution of $d_3$ is independent of $d_2$. However, the approach which we shall use will apply even if the distribution of $d_3$ depends on $d_2$. This is important in problems where there may be some postponement of the timing of demand. For example, it may be anticipated that the potential refrigerator buyers will buy in November or December. However, those buyers who failed to purchase in November, will affect the demand distribution for December.

**Example 4: A Class of Two-Stage Problems.**

In the Ferguson problem and in many supply problems the total costs may be divided into two parts: first the costs of assigning various resources to several destinations $j$ and second the costs (or lost revenues) incurred because of the failure of the total amounts $u_1, u_2, \ldots, u_n$ assigned to meet demands at various destinations in unknown amounts $d_1, d_2, \ldots, d_n$ respectively.

The special class of two-stage programming problems we are considering has the following structure.$^3$ For the first stage:

$$\sum_{j=1}^{n} x_{ij} = a_i \quad (x_{ij} \geq 0)$$

$$\sum_{i=1}^{m} b_{ij} x_{ij} = u_j$$

$^2$ No solution for this example will be given in this paper. For this case perhaps the simplest approach is through the techniques of dynamic programming; see R. Bellman [1].

$^3$ The remarks of this section apply if (6) and (7) are replaced more generally by $AX = a$, $BX = U$ where $X$ is the vector of activity levels in the first stage, $A$ and $B$ are given matrices, $a$ a given initial status vector, and $U = (u_1, u_2, \ldots, u_n)$. 
where \( x_{ij} \) represents the amount of \( i \)th resource assigned to the \( j \)th destination and \( b_{ij} \) represents the number of units of demand at destination \( j \) that can be satisfied by one unit of resource \( i \). For the second stage

\[
d_j = u_j + v_j - s_j \quad (j = 1, 2, \ldots, n)
\]

where \( v_j \) is the shortage of supply and \( s_j \) is the excess of supply.

The total cost function is assumed to be of the form

\[
C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} \alpha_j v_j
\]
i.e., depends linearly on the choice \( x_{ij} \) and on the shortages \( v_j \) (which depend on assignments \( u_j \) and the demands \( d_j \)).

Our objective will be to minimize total expected costs.\(^4\) Let \( \phi_j(u_j | d_j) \) be the minimum costs at a destination if the supply is \( u_j \) and the demand is \( d_j \). It is clear that

\[
\phi_j(u_j | d_j) = \begin{cases} 
\alpha_j(d_j - u_j) & \text{if } d_j \geq u_j \\
0 & \text{if } d_j < u_j
\end{cases}
\]

where \( \alpha_j \) is the coefficient of proportionality. We shall now give a result due to H. Scarf.

Theorem: The expected value of \( \phi_j(u_j | d_j) \), denoted by \( \phi_j(u_j) \) is a convex function of \( u_j \).

Proof: Let \( p(d_j) \) be the probability density of \( d_j \), then

\[
\phi_j(u_j) = \alpha_j \int_{u_j}^{+\infty} (x - u_j)p(x) \, dx
\]

\[
= \alpha_j \int_{u_j}^{+\infty} xp(x) \, dx - \alpha_j u_j \int_{u_j}^{+\infty} p(x) \, dx
\]

whence differentiating \( \phi(u) \)

\[
\phi_j'(u_j) = -\alpha_j \int_{u_j}^{+\infty} p(x) \, dx.
\]

It is clear that \( \phi_j'(u_j) \) is a non-decreasing function of \( u_j \) with \( \phi_j''(u_j) \geq 0 \) and that \( \phi_j(u_j) \) is convex. An alternative proof (due also to Scarf) is obtained by applying a lemma which we shall use later on.

Lemma: If \( \phi(x_1, x_2, \ldots, x_n | \theta) \) is a convex function over a fixed region \( \Omega \) for

\(^4\) Equation (8) should be viewed more generally then simply as a statement about the shortage and excess of supply. In fact, given any \( u_i \) and \( d_j \), there is an infinite range of possible values of \( v_j \) and \( s_j \) satisfying (8). For example, \( v_j \) might be interpreted as the amount obtained from some new source (perhaps at some premium price) and \( s_j \) the amount not used. When the cost form is as in (9), it becomes clear that in order for \( c \) to be a minimum, the values of \( v_j \) and \( s_j \) will have the more restrictive meaning above.

\(^5\) H. Markowitz in his analysis of portfolios considers the interrelation of the variance with the expected value. See [5].
every value of $\theta$, then any positive linear combination of such functions is also convex in $\Omega$.

In particular if $\theta$ is a random variable with probability density $p(\theta)$, then expected value of $\phi$

$$\phi(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{+\infty} \phi(x_1, x_2, \ldots, x_n|\theta)p(\theta) \, d\theta$$

is convex. For example from (10), $\phi(u_j | d_j)$, plotted below, is convex.

$$\phi(u_j|d_j) \quad \alpha_j \quad d_j$$

From the lemma the result readily follows that $\phi_j(u_j)$ is convex.

From the basic theorem the expected value of the objective function is

$$\text{Exp } C = \sum c_i x_i + \sum_{j=1}^{n} a_j \phi_j(u_j)$$

where $\phi_j(u_j)$ are convex functions. Thus the original problem has been reduced to minimizing (15) subject to (6), (7).

This permits application of a well-known device for approximating such a problem by a standard linear programming problem in the case the objective function can be represented by a sum of convex functions. See for example [3] or Charnes and Lemke, [2]. To do this one approximates the derivative of $\phi(u)$ in some sufficiently large range $0 \leq u \leq u_0$ by a step function

$${\phi}'(u)$$

involving $k$ steps where size of the $i^{th}$ base is $a_i$ and its height is $h_i$; where $h_1 \leq$
\[ h_2 \leq \cdots \leq h_k \text{ because } \phi \text{ is convex. An approximation for } \phi(u) \text{ is given by} \]

\[ \phi(u) = \phi(0) + \min \sum h_i \Delta_i \]

subject to

\[ u = \sum \Delta_i, \quad 0 \leq \Delta_i \leq a_i. \]

Indeed, it is fairly obvious that the approximation achieves its minimum by choosing \( \Delta_1 = a_1, \Delta_2 = a_2, \cdots \) until the cumulative sum of the \( \Delta_i \) exceeds \( u \) for some \( i = r; \Delta_r \) is then chosen as the value of the residual with all remaining \( \Delta_{r+i} = 0 \). In other words, we have approximated an integral by the sum of rectangular areas under the curve up to \( u \), i.e.,

\[ \phi(u) = \phi(0) + \int_0^u \phi'(x) \, dx = \sum h_i a_i + h_r \Delta_r. \]

The next step is to replace \( \phi(u) \) by \( \sum h_i \Delta_i \), \( u \) by \( \sum \Delta_i \) in the programming problem and add the restrictions \( 0 \leq \Delta_i \leq a_i \). If the objective is minimization of total costs, it will, of necessity, for whatever value of \( u = \sum \Delta_i \) and \( 0 \leq \Delta_i \leq a_i \), minimize \( \sum h_i \Delta_i \). Thus, this class of two-stage linear programming problems involving uncertainty can be reduced to a standard linear programming type problem. In addition, simplifying computational methods exist when variables have upper bounds such as \( \Delta_i \leq a_i \); see [3].

Example 5: The Two-Stage Problem with General Linear Structure.

We shall prove a general theorem on convexity for the two-stage problem that forms the inductive step for the multi-stage problem. We shall say a few words about the significance of this convexity later on. The assumed structure of the general\(^6\) two-stage model is

\[ b_1 = A_{11}X_1 \]

\[ b_2 = A_{12}X_1 + A_{22}X_2 \]

\[ C = \phi(X_1, X_2 | E_2) \]

where \( A_{ij} \) are known matrices, \( b_1 \) a known vector of initial inventories. For example

\[ a_i = \sum_{j=1}^n x_{ij} \quad \text{here } b_1 = (a_1, a_2, \cdots, a_m) \]

\[ d_j = \sum_{i=1}^n b_{ij} x_{ij} + v_j - s_j \quad \text{here } X_1 = (x_{11}, \cdots, x_{1n}, x_{21}, \cdots, x_{2n}, \cdots, x_{mn}) \]

\[ C = \sum \sum c_{ij} x_{ij} + \sum \alpha c_{ij} \quad \text{here } X_2 = (v_1, v_2, \cdots, v_n, s_1, s_2, \cdots, s_n) \]

\( b_2 \) an unknown vector whose components are determined by a chance mechanism.\(^7\)

\(^6\) A special case of the general model given in (20) is found in Example 4.

\(^7\) The chance mechanism may be the "market," the "weather."
(Mathematically, $E_2$ is a sample point drawn from a multidimensional sample space with known probability distribution; $X_1$ is the vector of nonnegative activity levels to be determined in the first stage, while $X_2$ is the vector of nonnegative activity levels for the second stage. It is assumed that whatever be the choice of $X_1$ satisfying the first-stage equations and whatever be the particular values of $b_2$ determined by chance, there exists at least one vector $X_2$ satisfying the second-stage equations. The total costs $C$ of the program are assumed to depend on the choice of $X_1$, $X_2$, and parametrically on $E_2$. The basic problem is to choose $X_1$ and later $X_2$ in the second stage such that the expected value of $C$ is a minimum.

Theorem: If $\phi(X_1, X_2 | E_2)$ is a convex function in $X_1$, $X_2$ whatever be $X_1$ in $\Omega_1$, i.e., satisfying the 1st stage restrictions and whatever be $X_2$ in $\Omega_2 = \Omega_2(X_1 | b_2)$, i.e., satisfying the 2nd stage restrictions given $b_2$ and $X_1$, then there exists a convex function $\phi_0(X_1)$ such that the optimal choice of $X_1$ subject to $b_1 = A_1X_1$ is found by minimizing $\phi_0(X_1)$ where

$$
\phi_0(X_1) = \text{Exp} \left[ \inf_{X_2 \in \Omega_2} \phi(X_1, X_2 | E_2) \right],
$$

$$
\text{Exp} \ C = \inf_{X_1 \in \Omega_1} \phi_0(X_1);
$$

the expectation ($\text{Exp}$) is taken with respect to the distribution of $E_2$ and the greatest lower bound ($\inf$) is taken with respect to all $X_2 = \Omega_2(X_1 | E_2)$.

Proof. In order to minimize the $\text{Exp} \phi_1(X_1, X_2 | E_2)$, it is clear that once $X_1$ has been selected, $E_2$ determined by chance, that $X_2$ must be selected so that $\phi(X_1, X_2 | E_2)$ is minimized for fixed $X_1$ and $E_2$. Thus, the costs for given $X_1$ and $E_2$ is given by

$$
\phi_1(X_1 | E_2) = \inf_{X_2 \in \Omega_2} \phi(X_1, X_2 | E_2).
$$

The expected costs for a given $X_1$ is then simply the expected value of $\phi_1(X_1 | E_2)$ and this we denote by $\phi_0(X_1)$. The optimal choice of $X_1$ to minimize expected costs $C$ is thus reduced to choosing $X_1$ so as to minimize $\phi_0(X_1)$. There remains only to establish the convexity property. We shall show first that $\phi_1(X_1 | E_2)$ for bounded $\phi_1$ is convex for $X_1$ in $\Omega_1$. If true, then applying the lemma, the result that $\phi_0(X_1)$ is convex readily follows. Let us suppose that $\phi_1(X_1 | E_2)$ is not convex, then there exist three points in $\Omega_1 : X_1', X_1'', X_1''' = \lambda X_1' + \mu X''$ ($\lambda + \mu = 1, 0 \leq \lambda \leq 1$) that violate the condition for convexity, i.e.,

$$
\lambda \phi_1(X_1' | E_2) + \mu \phi_1(X_1'' | E_2) < \phi_1(X_1''' | E_2)
$$

* The greatest lower bound instead of minimum is used to avoid the possibility that the minimum value is not attained for any admissible point $X_1 \in \Omega_2$ or $X_1 \in \Omega_1$. In case where the latter occurs, it should be understood that while there exists no $X_i$ where the minimum is attained, there exists $X_i$ for which values as close to minimum as desired are attained.

* This proof is along lines suggested by I. Glicksberg.
or
\[(24) \quad \lambda \phi_1(X_1' \mid E_2) + \mu \phi_1(X_1'' \mid E_2) = \phi_1(X_1''' \mid E_2) - \epsilon_0 \quad \epsilon_0 > 0.\]
For any \(\epsilon_0 > 0\), however, there exists \(X_1'\) and \(X_1''\) such that
\[(25) \quad \phi_1(X_1' \mid E_2) = \phi(X_1', X_2' \mid E_2) - \epsilon_1 \quad 0 \leq \epsilon_1 < \epsilon_0 \]
\[\phi_1(X_1'' \mid E_2) = \phi(X_1'', X_2'' \mid E_2) - \epsilon_2 \quad 0 \leq \epsilon_2 < \epsilon_0.\]
Setting \(X_1''' = \lambda X_1' + \mu X_1''\) we note because of the assumed linearity of the model \((20)\) that \((\lambda X_1' + \mu X_1'') \in \Omega_2(\lambda X_1' + \mu X_1'' \mid E_2)\) and hence by convexity of \(\phi\)
\[(26) \quad \lambda \phi(X_1', X_1' \mid E_2) + \mu \phi(X_1'', X_1'' \mid E_2) \geq \phi(X_1''', X_1''' \mid E_2)\]
whence by \((25)\)
\[(27) \quad \lambda \phi_1(X_1' \mid E_2) + \mu \phi_1(X_1'' \mid E_2) \geq \phi(X_1''', X_1''' \mid E_2) - \lambda \epsilon_1 - \mu \epsilon_2\]
and by \((24)\)
\[(28) \quad \phi_1(X_1''' \mid E_2) \geq \phi(X_1''', X_1''' \mid E_2) - \lambda \epsilon_1 - \mu \epsilon_2 + \epsilon_0 \quad (0 \leq \lambda \epsilon_1 + \mu \epsilon_2 < \epsilon_0)\]
which contradicts the assumption that \(\phi_1(X_1''' \mid E_2) = \text{Inf} \phi(X_1''', X_2 \mid E_2)\). The proof for unbounded \(\phi\) is omitted.

Example 5: The Multi-Stage Problem with General Linear Structure.

The structure assumed is
\[(29) \quad b_1 = A_{11}X_1 \]
\[b_2 = A_{12}X_1 + A_{12}X_2 \]
\[b_3 = A_{21}X_1 + A_{22}X_2 + A_{32}X_2 \]
\[b_4 = A_{41}X_1 + A_{42}X_2 + A_{43}X_3 + A_{44}X_4\]
\[\text{.........................} \]
\[b_m = A_{m1}X_1 + A_{m2}X_2 + A_{m3}X_3 + \cdots \cdots + A_{mm}X_m \]
\[C = \phi(X_1, X_2, \cdots, X_m \mid E_2, E_3, \cdots, E_m)\]

where \(b_1\) is a known vector; \(b_i\) is a chance vector \((i = 2, \cdots, m)\) whose components are functions of a point \(E_i\) drawn from a known multi-dimensional distribution; \(A_{ij}\) are known matrices. The sequence of decisions is as follows: \(X_1\), the vector of nonnegative activity levels in the 1st stage, is chosen so as to satisfy the first stage restrictions \(b_1 = A_{11}X_1\); the values of components of \(b_2\) are chosen by determining \(E_2\); \(X_2\) is chosen to satisfy the 2nd stage restrictions \(b_2 = A_{21}X_1 + A_{22}X_2\), etc. iteratively for the third and higher stages. It is further assumed that:

(1) The components of \(X_j\) are nonnegative;

(2) There exists at least one \(X_j\) satisfying the \(j\)th stage restraints, whatever be
the choice of $X_1, X_2, \ldots, X_{j-1}$ satisfying the earlier restraints or the outcomes $b_1, b_2, \ldots, b_m$.

(3) The total cost $C$ is a convex function in $X_1, \ldots, X_m$ which depends on the values of the sample points $E_2, E_3, \ldots E_m$.

**Theorem:** An equivalent $(m - 1)$ stage programming problem with a convex pay-off function can be obtained by dropping the $m$th stage restrictions and replacing the convex cost function $\phi$ by

$$\phi_{m-1}(X_1, X_2, \ldots, X_{m-1} | E_2, \ldots, E_{m-1}) = \text{Exp} \inf_{E_m} \phi(X_1, X_2, \ldots, X_m | E_2, \ldots, E_m)$$

where $\Omega_m$ is the set of possible $X_m$ that satisfy the $m$th stage restrictions.

Since the proof of the above theorem is identical to the two-stage case no details will be given. The fact that a cost function for the $(m - 1)$ stage can be obtained from the $m$th stage is simply a consequence that optimal behavior for the $m$th stage is well defined, i.e., given any state, e.g., $(X_1, X_2, \ldots, X_{m-1})$, at the beginning of this stage, the best possible actions can be determined and the minimum expected cost evaluated. This is a standard technique in “dynamic programming.” For the reader interested in methods built around this approach the reader is referred to R. Bellman’s book on dynamic programming [1].

While the existence of convex functions has been demonstrated that permit reduction of an $m$-stage problem to equivalent $m-1, m-2, \ldots, 1$-stage problems, it appears hopeless that such functions can be computed except in very simple cases. The convexity theorem was demonstrated not as a solution to an $m$-stage problem but only in the hope that it will aid in the development of an efficient computational theory for such models. It should be remembered that any procedure that yields a local optimum will be a true optimum if the function is convex. This is important because multi-dimensional problems in which non-convex functions are defined over non-convex domains lead as a rule to local optimum and an almost hopeless task, computationally, of exploring other parts of the domain for the other extremes.

**Solution for Example 2:** *Shipping to an Outlet to Meet an Uncertain Demand.*

Let us consider the two-stage case given earlier (4). It is clear that, if supply exceeds demand ($x_{11} > d_2$), that $x_{11} = 0$ gives minimum costs and, if $x_{11} \leq d_2$, that $x_{11} = d_2 - x_{11}$ gives minimum costs. Hence

$$\text{Min } \phi = \begin{cases} x_{11} & \text{if } x_{11} > d_2 \\ x_{11} + 2(d_2 - x_{11}) & \text{if } x_{11} \leq d_2. \end{cases}$$

Since $d_2$ is assumed to be uniformly distributed between 70 and 80

$$\text{Exp} [\text{Min } \phi] = \begin{cases} -x_{11} + 150 & \text{if } x_{11} \leq 70 \\ 77.5 + \frac{1}{10}(75 - x_{11})^2 & \text{if } 70 < x_{11} \leq 80 \\ x_{11} & \text{if } 80 \leq x_{11}. \end{cases}$$
This function is clearly convex and attains its minimum 77.5, which is the expected cost, at \( x_{11} = 75 \). Since \( x_{11} = 75 \) is in the range of possible values of \( x_{11} \) as determined by \( 100 = x_{11} + x_{12} \) this is clearly the optimal shipment. In this case it pays to ship \( x_{11} = d_2 = 75 \), the expected demand.

It can be shown by simple examples that one cannot replace, in general, the chance vectors \( b_i \) by \( \bar{b}_i \), the vector of expected values of the components of \( b_i \). Nevertheless, this procedure, which is quite common, probably provides an excellent starting solution for any improvement technique that might be devised. For example, in the problem of Ferguson (application of Example 4), using as a start the solution based on expected values of demand, it was an easy matter to improve the solution to an optimal one whose expected costs were 15% less.

Solution for Example 5: The General Two-Stage Case.

When the number of possibilities for the chance vector \( b_i \) is \( b_2^{(1)}, b_2^{(2)}, \ldots, b_2^{(k)} \) with probabilities \( p_1, p_2, \ldots, p_k \), \( (\sum p_i = 1) \), it is not difficult to obtain a direct linear programming solution for small \( k \), say \( k = 3 \). Since this type of structure is very special, it appears likely that techniques can be developed to handle large \( k \). For \( k = 3 \), the problem is equivalent to determining vectors \( X_1 \) and vectors \( X_2^{(1)}, X_2^{(2)}, X_2^{(3)} \) such that

\[
\begin{align*}
  b_1 &= A_{11}X_1 \\
  b_2^{(1)} &= A_{21}X_1 + A_{22}X_2^{(1)} \\
  b_2^{(2)} &= A_{22}X_1 + A_{23}X_2^{(2)} \\
  b_2^{(3)} &= A_{22}X_1 + A_{23}X_2^{(3)} \\
\end{align*}
\]

(33) \[ \text{Exp } C = \gamma_1X_1 + p_1\gamma_2X_2^{(1)} + p_2\gamma_2X_2^{(2)} + p_3\gamma_2X_2^{(3)} = \text{Min} \]

where for simplicity we have assumed a linear objective function.

References
