Optimal Hedging of Interest Rate Exposure Given Credit Correlation

Ray Chen, Abhay Subramanian, Xiao Tang, Michael Turrin

Stanford University, MS&E 444
1. Introduction

Interest rate risk arises from the variation that occurs in interest rates over time. Both fixed income and credit instruments are susceptible to such risk. A traditional “Greeks” based approach to hedging away interest rate risk involves adding the DV01 values across the portfolio and shorting treasuries or trading interest rate swaps until the total DV01 equals zero. The problem with this approach lies in the fact that it fails to account for the correlation between treasury bonds and credit instruments, in particular the default probabilities of the asset underlying CDSs.

Figure 1 shows a scatter plot of the CDX Credit Spread against the 10-year Treasury bill yields. The color scale in the plot is used to represent time. From this plot come two important observations. First, there is very little mixture of colors in this plot. Instead, each color appears to cluster within one particular place on the graph. From this, one can infer that the correlation between the CDX and Treasury bill changes over time. Looking further, we observe that this change in correlation is directly related to the macroeconomic state. The yellow-orange region in the upper left hand of this plot represents a stressed financial environment during which the correlation becomes sharply negative, while the reddish and the bluish portion of this plot represents a more normal period in which this correlation is much less negative and may even take on slightly positive values.

Figure 1: CDX vs. 10-Year Treasury Bill, 2005-11
Another way of observing the correlation between the CDX and 10-Year Treasury bill is to look at the time series plots of these two quantities between 2005 and 2011, as shown in Figure 2. From this figure, we observe a negative correlation, which is exceptionally strong during 2008 and 2009. During this time period, the yield of the Treasury bill falls while the CDX level spikes upward at what appears to be the exact same time.

**Figure 2: 10 Year Treasury Note Yield & CDX**

2. **Hedging**

The main purpose of our project is to improve the performance of a portfolio of fixed income and credit instruments by developing a hedging strategy that is able to account for the observed correlation between credit instruments and interest rates. For our hedging strategy to be effective, it is important that it be able to account for the observed time varying nature of this correlation.

Theoretical models look at the sensitivities of a portfolio to changes in the interest rate and CDX separately, but fail to account for this correlation. We hope to improve upon these models by utilizing a dynamic hedging strategy, which takes correlation into account. Our strategy is as follows:

Current pricing models allow us to compute the theoretical exposures of our portfolio to both movements in interest rates and credit markets. Assuming that credit (in particular, the default probabilities) and interest rates are not correlated, these theoretical exposures can be expressed (and calculated) using partial derivatives as:
Interest rate exposure = \( \frac{\partial V}{\partial I} \);

CDX exposure = \( \frac{\partial V}{\partial C} \),

where \( V \) denotes the value of the portfolio, \( C \) denotes the value of the CDX index and \( I \) denotes the (10 year) US Treasury yield.

If these two are indeed not correlated, then hedging of interest rate exposure simply entails buying (or selling) enough US Treasury Notes to neutralize the interest rate exposure, \( \partial V/\partial I \).

However, it is our observation that these two are indeed correlated and, as a result, we use an additional term to account for this correlation. From this, the interest rate exposure of our portfolio becomes:

\[
\text{Interest rate exposure} = \frac{\partial V}{\partial I} + \frac{\partial V}{\partial C} \cdot \frac{\partial C}{\partial I}
\]

Thus, one approach to improving upon the naïve hedging strategy, which does not take this correlation into account, is to first estimate \( \partial C/\partial I \) using historical data, and then use this estimate along with the values given by theoretical models for \( \partial V/\partial C \) and \( \partial V/\partial I \) to calculate the adjusted interest rate exposure which guides our hedging decisions. The following sections will describe a variety of methods and models we used to estimate \( \partial C/\partial I \). Using these estimates, we will compare the results of our hedging strategy to those of the naïve approach across various performance measures for a portfolio of fixed income and credit instruments with a $50 million initial value.

### 3. Markov Regime-Switching Models

The first model we chose to investigate for implementing our hedging strategy was the Hidden Markov Model (HMM). In this context, a Hidden Markov Model assumes that the state of the economy evolves according to a Markov chain model whose states are not directly observable. The characteristics (like the probability distribution) of other observable data depend on the current state of this “hidden” Markov chain. Fitting such a model to some observed data set entails specifying the size of the state space of the hidden Markov chain and then estimating the transition probabilities of this Markov chain and the parameters of the distribution of the observed data in each state of the Markov chain.

To start, we used the HMM model to classify the available data into a number of states. We then conducted a regression of the CDX index on the 10-year Treasury
yields using the observations classified into each state, after deleting any outliers that do not appear to belong to either state. Then, we used the regression coefficients (betas) as estimates of $\partial C/\partial I$ in each state. We do this under the belief that a linear relationship between CDX and interest rates in each state is a reasonable assumption. The HMM estimator was implemented using both univariate and multivariate data. The univariate data was the time series of our estimates of $\partial C/\partial I$ over 60-day trailing windows. The multivariate data used was the log-differenced time series of the CDX and 10-year Treasury yields.

Our initial method of assessing these models was to compare the classification of states over time to what we would expect given our knowledge of historical data. We expect the HMM to classify our data into a “normal” state for the period from 2005 through mid-2007. After this we expect our data to be classified into a “stressed” period. Next, we look at the regression coefficients to ensure that a good separation of values for the two states is achieved and can be used in our hedging strategy. Our results are as follows:

First, we analyze the results provided by the two-state univariate Hidden Markov Model. This model classifies the time series of our $\partial C/\partial I$ estimates into two unobservable states, in each of which this quantity is assumed to follow a different normal distribution. Figure 3 provides a plot of the estimated probabilities of being in each state across time. We observe that State 1 is the “normal” state, which occurs until mid-2007, after which the “stressed” state occurs through the end of 2008. Table 1 provides the number of days the HMM spends in each state along with the regression coefficients computed for each state.

Figure 3: Univariate 2-state HMM - State Probabilities

Table 1: State Classification and $\partial C/\partial I$ of Univariate Two State HMM

<table>
<thead>
<tr>
<th></th>
<th>State 1 (Stressed)</th>
<th>State 2 (Normal)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Days in State</td>
<td>433</td>
<td>815</td>
</tr>
<tr>
<td>% Time in State</td>
<td>34.7</td>
<td>65.3</td>
</tr>
<tr>
<td>$\partial C/\partial I$</td>
<td>-32.63</td>
<td>-13.85</td>
</tr>
</tbody>
</table>

Next, we analyze the multivariate case, using log-return data for the 10-year Treasury bill and the CDX IG credit spread. Here, we fit a model in which the underlying Markov chain has two (or three) unobservable states and, in each of
these states, we assume that the multivariate observations come from a different first order autoregressive model (AR(1)) with normally distributed noise. The formal specification of an AR(1) model is:

\[ X_t = c + \phi X_{t-1} + \epsilon \]

Thus, our model states that today's observation is an affine function of yesterday's observation with an additive, normally distributed noise where the parameters \( c \) and \( \phi \) depend on the current state of the hidden Markov chain.

The results of our state classification and subsequent regression estimates of \( \partial C/\partial I \) in each state are provided below.

**Figure 4: State Classification, Multivariate 2-State HMM**

| Table 2: State Classification and \( \partial C/\partial I \) of Multivariate Two State HMM |
|-----------------------------------------------|-------------------------------|
| Days in State                  | State 1 (Stressed) | State 2 (Normal) |
| % Time in State           | 632                  | 674              |
| \( \partial C/\partial I \)   | -74.5744             | -43.0022         |

We observe that although the multivariate HMM does a similar job of grouping the data into states across time, it spends a longer percentage of time in the stressed environment.

Additionally, we look at a three-state multivariate HMM. By providing an additional state, it is our hope that this model will achieve better separation between “normal” and “stressed” states by creating an intermediate third state. Unfortunately, this does not occur and instead we observe a “normal” state and two similar “stressed” states. The model fluctuates between these two “stressed” states very frequently from mid-2007 on. The results of the state classification are provided in Figure 5 below.
Moving forward, we use the HMM fit to first predict the current state of the hidden Markov chain and then use the estimated (via regression) value of $\frac{\partial C}{\partial I}$ in the predicted current state as our current estimate of this quantity. With these estimates, we are then able to calculate our portfolio’s modified interest rate exposure, taking correlation into account, and determine how to hedge our portfolio. The results of this hedging strategy for the HMM and subsequently described models are provided in the “Results” section of this report.

4. Gaussian Mixture Model

Another statistical model we can look at using a state classification perspective is a Gaussian Mixture Model. This model uses unsupervised learning to group the data into a number of components which can be interpreted in the same way as the states of the HMM model discussed earlier. In this case, the data is the univariate time series of estimates of $\frac{\partial \log C}{\partial \log I}$ formed using regression with trailing 60-day windows. We then proceeded to fit a two component Gaussian mixture distribution to this data, constraining the variance of both components to be the same. From this, the model is able to look at the data as a mixture of two normal distributions and classify the data into one of these two “states.”

Figure 6 shows the results of the Gaussian Mixture state classification. Much like the HMM models previously described, this model also groups most of the data occurring prior to mid-2007 into a normal state (State 1) and the data from mid-2007 until 2009 into a stressed state (State 2). It is observed that the stressed period from the Gaussian model does not occur for as long as was previously observed using hidden Markov models.
Once the Gaussian Mixture Model has classified the data into two components/states, we can look at the values of our estimate for $\partial \log C / \partial \log I$ plotted over time for each of these states. Figure 7 provides this plot, in which the blue line represents a “normal” financial environment and the red line represents a “stressed” environment.

**Figure 7: Gaussian Mixture, $\partial \log C / \partial \log I$ vs. Time for 2 States**
We observe a separation of values between the two states. We also observe that the normal state occurs for significantly longer than the stressed financial state. The results of using this mixture model to estimate $\partial C/\partial I$ on our portfolio’s performance are provided in the results section of this report.

5. Function Fitting

Another way to obtain estimates of $\partial C/\partial I$ is to fit a deterministic function to the data. In this approach, we look at the CDX index level as a function of the interest rate. We fit a function to the observed relationship between these two quantities. The slope of this function at the observed value of the interest rate then serves as an estimate for $\partial C/\partial I$ at that time. Given these estimates, we are then able to again compute the modified interest rate exposure of our portfolio and determine the appropriate amount to hedge.

Figures 8 and 9 provide scatter plots of the CDX vs. Treasury Yield on a logarithmic scale. The red line in Figure 10 shows the results of fitting a linear function to this data. Similarly, the line in Figure 11 shows the results using a quadratic fit using polynomial regression.

Figure 8: Fitting a Linear Function to CDX vs. Treasury Yield
In addition to these methods, one can also estimate the relationship between these two quantities using a nearest neighbor local regression model using the T-bill as a predictor for the CDX level. We tried neighborhood sizes of 1, 2, 3..., 10 months of data and used 10-fold cross-validation to select the best model over a chosen training set. The model chosen in this fashion was the one that used a 5 month neighborhood. Then, the strategy we used was to observe the interest rate on a particular day in the test set and regress the CDX levels on the observed interest rates using the past 5 months’ worth of data when the interest rates were closest to the current level.

6. Mean Reverting Jump Diffusion Model

In observing the behavior of our trailing 60-day window estimates for \( \partial C/\partial I \), we observe significant spikes or “jumps” in the data. Figure 10 shows a time series plot of these estimates. We see that although state classification techniques are able to capture some of the large fluctuations in this data, perhaps another type of model may be able to additionally capture the smaller, possibly mean-reverting fluctuations. But the few large jumps that we observe in Figure 10 resulted in poor fits when estimating pure diffusion models using this univariate time series.
By using a mixture of mean-reversion, diffusion, and jump terms, the mean reverting jump diffusion model allows us to account for several of the observations we have made regarding the data. We observe that the data has a tendency to revert to a mean value, but that this mean value changes depending on the financial environment we are in. The jump diffusion model allows us to account for the large changes, which occur when we move from a normal to a stressed financial environment. The equation below represents makes precise the jump diffusion model we use for \( X_t \), the time series of estimated \( \partial C/\partial I \).

\[
dx_t = (\alpha - \beta X_t)dt + \sigma dW_t + Z_t dN_t
\]

The first term on the right hand side accounts for the mean-reverting behavior, the second term accounts for diffusion, and the third term accounts for jumps in the data – we assume that the jump sizes \( Z_t \) are independent normal random variables. Fitting this model to the data provides yet another strategy for estimating the interest rate exposure of our portfolio – the one step ahead forecast is the mean of a two component Gaussian mixture, one component representing the absence of a jump and the other representing the presence of one.

7. Pricing Model

Using literature, we find that the most commonly used model for interest rates and default intensity rates is the Cox-Ingersoll-Ross model. It specifies that the instantaneous interest rate and intensity rate follow the stochastic differential equation:

\[
dY_t = \kappa(\theta - Y_t)dt + \sigma \sqrt{Y_t}dW_t
\]
The drift factor $\kappa(\theta - Y_t)$ ensures the process reverts towards the long run value $\theta$, with the speed of mean reversion governed by $\kappa$.

We tried to link the interest rate model and the default intensity model together by adding a latent factor:

$$\lambda_t = r_t + X_t, \text{ where } X_t \text{ is also a CIR process}$$

In this case, we have implicitly required that the speed of mean reversion and volatility be the same for $\lambda_t$, $r_t$, and $X_t$.

We first calibrated the model parameters of the short rate, $r_t$ to the yield curve using a standard least squares fitting technique.

**Figure 11: Calibration of the Yield Curve**

Then, we fix $\kappa$ and $\sigma$, and calibrate the intensity rate $\lambda_t$ to the CDX spread value. Finally, the sensitivity $\partial C/\partial I$ is calculated by:

$$\frac{\partial C}{\partial I} = \frac{\partial C}{\partial \kappa} \frac{\partial \kappa}{\partial I} + \frac{\partial C}{\partial \sigma} \frac{\partial \sigma}{\partial I}$$

where all partial derivatives are estimated using the finite differences.

The performance of the pricing model is not as good as expected. It performs even worse than the naïve strategy using a weekly hedging basis. A closer look reveals that the model could not provide the negative correlation between interest rate and
CDX that we observed empirically since $\lambda_t$ and $r_t$ are always positively correlated according to this model.

**Figure 12: Calibrated Parameters**

![Graph showing calibrated parameters over time]

**8. Summary of Results**

Using each of the models described above, we are able to estimate the modified interest rate exposure of our portfolio. For our purposes, this portfolio consists of fixed income and credit instruments. We assume that the portfolio has an initial value of $50$ million and that the theoretical exposures $\partial V/\partial I$ and $\partial V/\partial C$ are equal to $-50,000$ and $-150,000$ respectively. Using these values and our estimates of $\partial C/\partial I$, we analyze the performance of our portfolio across a variety of performance measures. We use
- Volatility of daily returns,
- Downside (Negative) volatility of daily returns,
- 95th percentile of negative daily returns (i.e. 95% Value-at-Risk), and
- Sharpe ratios (ratio of annualized mean return to annualized volatility of returns), computed using volatility as well as downside volatility
to determine which models provide the most effective results. The naïve hedging strategy, which does not account for $\partial C/\partial I$ (or effectively assumes that this is identically 0), is used as a benchmark against which each of our models is compared. We experimented with a variety of hedging frequencies and training/testing data sets to verify the consistency of our findings. Results gathered using 80% of our data as training data and 20% as testing data are representative of our findings. Tables 3 and 4 provide a summary of our results using daily and weekly hedging frequencies.
The first column of these tables gives the results of the naïve approach. The subsequent columns provide results using the two-state multivariate Hidden Markov model, Gaussian mixture model, quadratic function fitting model, nearest neighbor local regression model, and mean reverting jump diffusion model. From the daily hedging results, we see that all models, except the local regression, outperform the naïve approach on most measures. In particular, we observe the jump diffusion model is superior to all other models across each of these performance measures.

Table 3: Summary of Results using 80% Training data and Daily Hedging

<table>
<thead>
<tr>
<th></th>
<th>Naïve</th>
<th>HMM</th>
<th>Gaussian Fit</th>
<th>Powerlaw Fit</th>
<th>Local Regression</th>
<th>MRJD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily Volatility</td>
<td>1.120</td>
<td>1.128</td>
<td>0.915</td>
<td>1.096</td>
<td>1.528</td>
<td>0.902</td>
</tr>
<tr>
<td>Daily downside volatility</td>
<td>0.936</td>
<td>0.884</td>
<td>0.698</td>
<td>0.762</td>
<td>1.348</td>
<td>0.702</td>
</tr>
<tr>
<td>95% VaR</td>
<td>0.016</td>
<td>0.021</td>
<td>0.014</td>
<td>0.019</td>
<td>0.026</td>
<td>0.014</td>
</tr>
<tr>
<td>Annualized Sharpe ratio (Vol.)</td>
<td>-0.035</td>
<td>0.470</td>
<td>0.251</td>
<td>0.536</td>
<td>-0.226</td>
<td>0.615</td>
</tr>
<tr>
<td>Annualized Sharpe ratio (Neg. Vol.)</td>
<td>-0.041</td>
<td>0.599</td>
<td>0.329</td>
<td>0.770</td>
<td>-0.256</td>
<td>0.789</td>
</tr>
</tbody>
</table>

Table 4: Summary of Results using 80% Training data and Weekly Hedging

<table>
<thead>
<tr>
<th></th>
<th>Naïve</th>
<th>HMM</th>
<th>Gaussian Fit</th>
<th>Powerlaw Fit</th>
<th>Local Regression</th>
<th>MRJD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily Volatility</td>
<td>1.120</td>
<td>1.044</td>
<td>0.910</td>
<td>1.099</td>
<td>1.371</td>
<td>0.912</td>
</tr>
<tr>
<td>Daily downside volatility</td>
<td>0.936</td>
<td>0.780</td>
<td>0.718</td>
<td>0.765</td>
<td>1.144</td>
<td>0.705</td>
</tr>
<tr>
<td>95% VaR</td>
<td>0.016</td>
<td>0.017</td>
<td>0.014</td>
<td>0.018</td>
<td>0.024</td>
<td>0.015</td>
</tr>
<tr>
<td>Annualized Sharpe ratio (Vol.)</td>
<td>-0.035</td>
<td>0.368</td>
<td>0.408</td>
<td>0.534</td>
<td>0.136</td>
<td>0.620</td>
</tr>
<tr>
<td>Annualized Sharpe ratio (Neg. Vol.)</td>
<td>-0.041</td>
<td>0.493</td>
<td>0.517</td>
<td>0.767</td>
<td>0.163</td>
<td>0.802</td>
</tr>
</tbody>
</table>

Comparing the results generated using weekly hedging rather than daily hedging, we see minimal impact in our results. We note the improved performance of the hidden Markov and Gaussian mixture models. Again, all models other than local regression outperform the naïve approach on most measures. With weekly hedging, the jump diffusion and Gaussian mixture model provide the best results of our models.

9. Conclusions and Future Work

From our results, we see that by taking into account the observed correlation between the CDX and 10-year Treasury bill and the change in this correlation over time, our strategy for hedging away interest rate exposure given a portfolio of fixed income and credit instruments outperforms the naïve Greeks-based approach. In particular, we observe that the Gaussian mixture model and mean-reverting jump
diffusion model provide the best results with respect to portfolio's volatility, VaR and Sharpe ratio. Additionally, we find that hedging our portfolio on a weekly basis allows for optimal results taking into account the popular practitioners' view that the CDX index tends to contain a lot of noise and loses signal when observed on the daily frequency.

Moving forward, it may prove interesting to apply a combination of these models to this problem – in particular, a combination of jump diffusion and the HMM type models – to determine whether significant improvement occurs when using an optimal mix of these models.

A pure theoretical asset pricing based approach to hedging using reduced form models for interest rates and default intensities is another possible extension. In particular, if one can formulate a CDS pricing model that incorporates a correlation between the risk-free interest rates and default intensities (instead of the standard assumption of independence between the two processes), then the theoretical value of "rho" (or the interest rate sensitivity) obtained using such a model can be a better guide for determining optimal hedge ratios.

References
