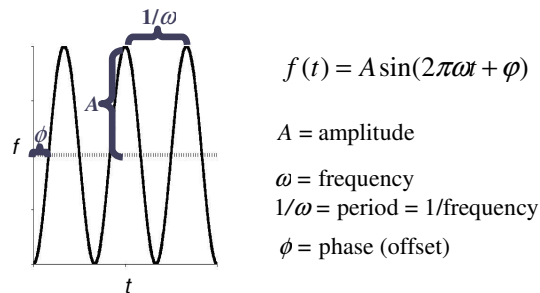


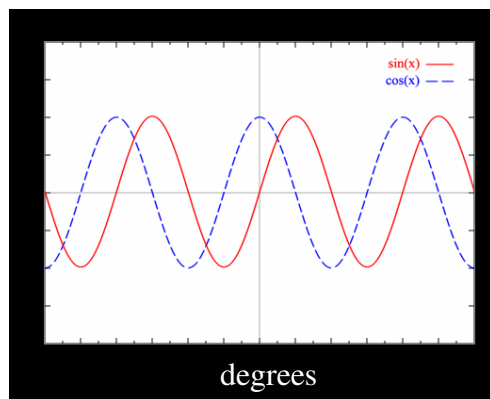
Fourier Analysis

Fourier analysis follows from *Fourier's theorem*, which states that every function can be completely expressed as a sum of sines and cosines of various amplitudes and frequencies. This is a pretty impressive assertion – no matter what the shape of a function, and how little it looks like a sine wave, it can be rewritten as a sum of sines and cosines. The *Fourier series* tells you the amplitude and frequency of the sines and cosines that you should add up to recreate your original function.

Before getting into the details of Fourier series, it may help to briefly review the terms associated with a sine wave with the figure below.



A cosine wave is just a sine wave shifted in phase by 90° ($\phi=90^\circ$).

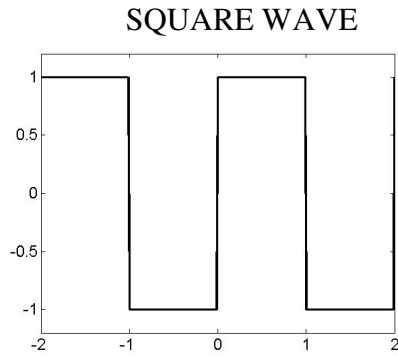


Cosine functions are **even** functions while sine wave are **odd** functions. An even function is symmetric with respect to the y-axis, meaning that its graph remains unchanged after reflection about the y-axis. On the other hand, an odd function is symmetric with respect to the origin, meaning that its graph remains unchanged after rotation of 180 degrees about the origin.

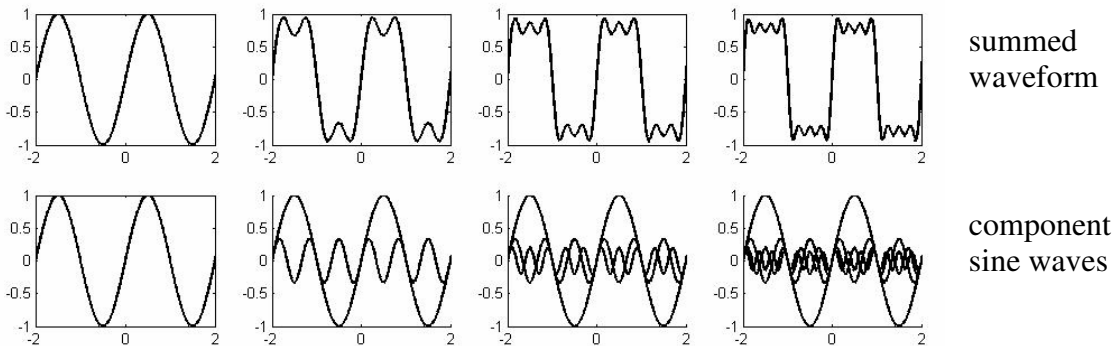
In the language of linear algebra, Fourier's theorem states that sine waves and cosine wave create a complete **basis set** that spans all possible functions. Sines and cosines are in fact **independent** (and also **orthogonal**) -- there is no way to add up cosine waves to create a sine wave.

A nice example of Fourier's Theorem is the creation of a square wave by summing the appropriate component sine waves. Like a sine or a cosine wave, a square

wave is a periodic function. Unlike those other functions, a square wave has sharp corners at 90° angles (see figure below).



How could we create a square wave out of sines and cosines? In the figure below, the top panels represents a waveform and the bottom panel represents the sine waves that are added together to form the waveform. Each column contains one more component sine wave than the previous panel. As you can see, as you add up more sine waves, the resulting waveform starts to look more and more like a square wave. An infinite sum of the appropriately chosen sine waves would lead to a perfect square wave.



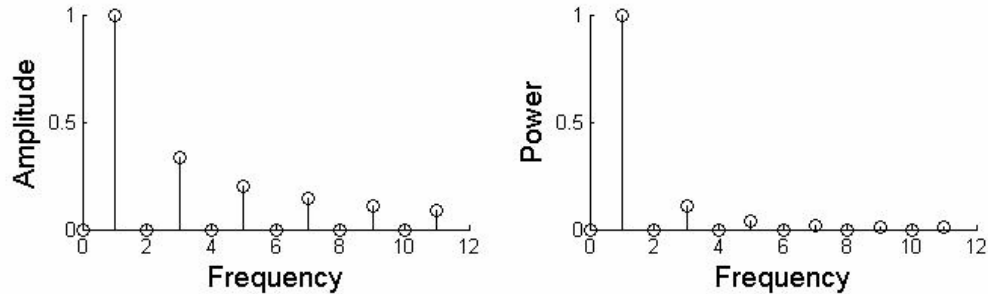
How did I know which sine wave should be summed to create the square wave? The Fourier series is used to figure out which sine and cosine waves should be summed, at what amplitude, to create a periodic waveform of interest. This is the Fourier series for a square wave:

$$\begin{aligned}
 \text{square}(x) &= \sin \pi x + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \frac{\sin 7\pi x}{7} + \dots \\
 &= \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{2k+1}
 \end{aligned}$$

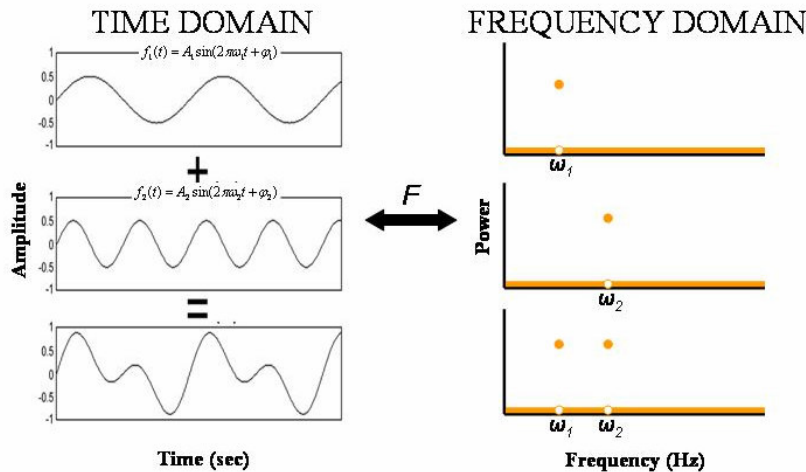
Notice that only sines, and not cosines, contribute to creating the square wave. That's because the square wave that I've drawn is an odd function, just like a sine wave. If we shifted the square wave by 90°, we would have summed cosines to create it instead.

Once we know the Fourier series for a square wave, the square wave can easily be expressed in the **frequency domain**. This involves plotting the amplitude or the power spectrum. The amplitude spectrum is just the amplitude of the sine/cosine at each

frequency, while the power spectrum is the square of the amplitude spectrum. Based on the equation for the square wave above, the amplitude is 1 for frequency of 1 ($\omega=1$), 1/3 for frequency of 3 ($\omega=3$), and so on. The amplitude/power is zero at even frequencies for this square wave example.



The figure below is another simple example of plotting the same signal in both the time domain and frequency domain. The sine waves represented by the top two rows are summed to create the waveform in the bottom row. The right column shows how much power is in each frequency (“power spectrum”). Notice that the power is zero at most frequencies. This is because the waveforms in this example are composed of either 1 or 2 sine waves, so most frequencies are not contributing any power to the signal.



The extension of a Fourier series for a non-periodic function is known as the **Fourier transform**. When calculating the Fourier transform, rather than decomposing a signal in terms of sines and cosines, people often use complex exponentials. They can be a little easier to interpret, although they are mathematically equivalent. A complex exponential is defined as $Ae^{i\phi}$, where $i^2=-1$ (i is the “imaginary” number), A is the amplitude, and ϕ is the phase. A waveform can be decomposed in terms of complex exponentials rather than sines and cosines because of Euler’s Theorem, which highlights the surprisingly close relationship between a complex exponential and sines/cosines.

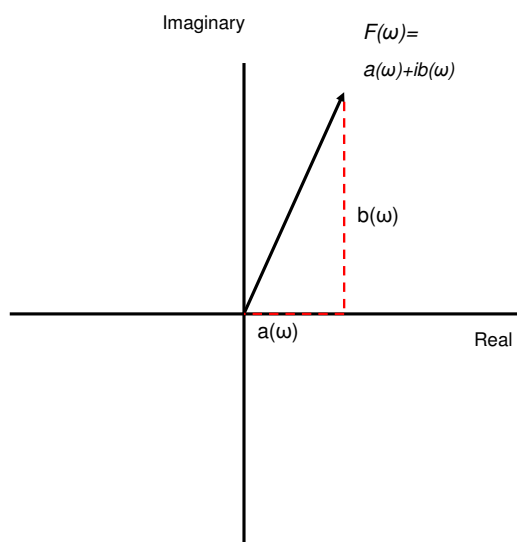
EULER’S THEOREM: $e^{i\phi} = \cos \phi + i \sin \phi$

The Fourier transform allows you to write any function ($f(t)$) as the integral (sum) across frequencies of complex exponentials of different amplitudes and phases ($F(\omega)$). $f(t)$ is often called the “time domain” representation while $F(\omega)$ is called the “frequency domain representation.” The key thing to understand about Fourier transforms is that these two representations are different ways of expressing the same information. The formula for a Fourier transform is below:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$F(\omega) = A(\omega) e^{i\phi(\omega)}$$

When you use a computer to take a Fourier transform of a function $f(t)$, the computer will return the complex number $F(\omega)$, which you should think of as a vector in the complex plane, as plotted below. The complex plane is a 2-D plane with real numbers along the x-axis and imaginary numbers along the y-axis.



The easiest way to interpret $F(\omega)$ is by calculating the amplitude (A) and the phase (ϕ) at each frequency. The amplitude (or “magnitude”) tells you how much signal there is at a given frequency, whereas the phase tells you the delay of the signal at a given frequency. Based on the geometric interpretation of a complex exponential, you can calculate the amplitude and phase of the complex exponential using some high school trigonometry. For a complex exponential $F(\omega) = a(\omega) + ib(\omega)$,

1) The amplitude, which is the length of the vector, can be calculated using the Pythagorean theorem:

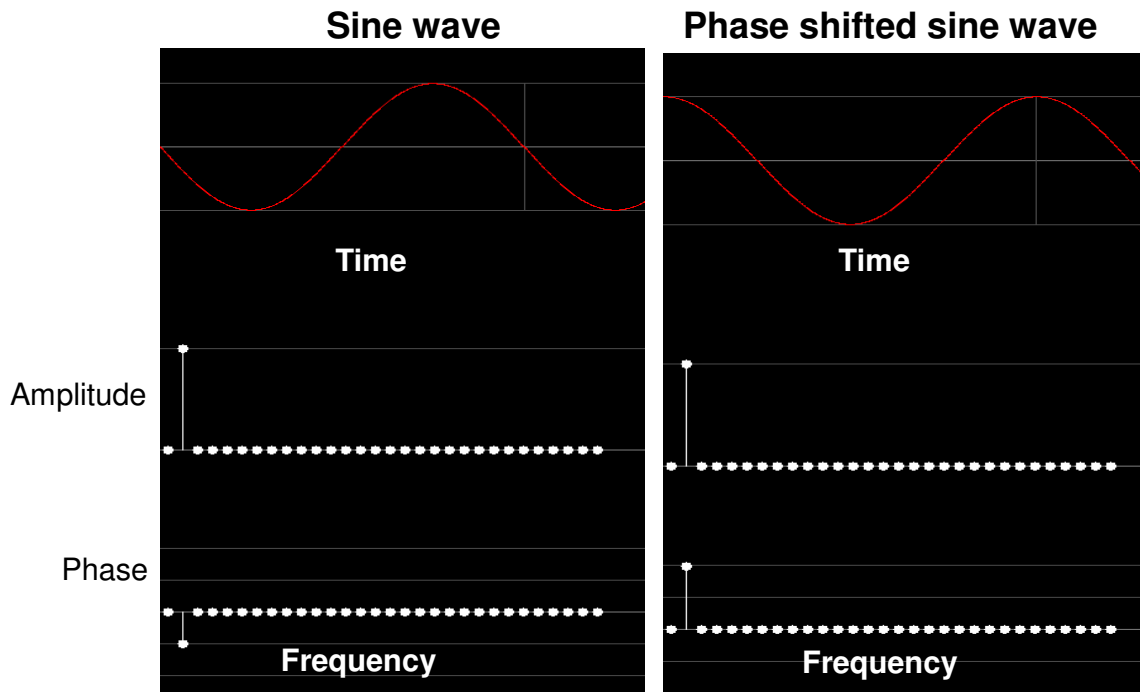
$$A = |F(\omega)| = \sqrt{a(\omega)^2 + b(\omega)^2}$$

2) The phase ($\angle F(\omega)$), which is the angle of the vector from the x-axis, can be calculated based on the arctangent – the angle whose tangent is $\frac{b(\omega)}{a(\omega)}$.

$$\phi = \angle F(\omega) = \tan^{-1} \frac{b(\omega)}{a(\omega)}$$

Once you know the complex exponential $F(\omega)$, you can transform back into the time domain by calculating the **inverse Fourier transform**, and recover the original function $f(t)$.

Below, you can see the Fourier Transform for a sine wave and a phase-shifted sine wave. Notice that the amplitude is identical for these two conditions, but the phase is different, reflecting the fact that the two signals are simply phase shifted (i.e. spatially offset) versions of each other.



So, why would you want to take a Fourier transform? Fourier analysis can be very useful for two main reasons.

1. *Many calculations are simpler in the frequency domain than the time domain.*
 - For example: filtering (convolving) becomes trivial in the frequency domain. We'll talk about this next chapter.
2. *Many neural processes can be described more effectively in the frequency domain.*
 - For example: The cochlea transforms a time domain signal (the sound's waveform) into a frequency domain signal. The strength of the response in the auditory nerve fiber tuned to a particular frequency reflects the amplitude of the sound's waveform at that frequency. In other words, the auditory system takes a Fourier transform of the incoming signal, decomposing the sound into amplitudes as a function of frequency.

- Many brain regions have oscillations of a particular frequency that can be easily characterized with Fourier analysis.