The expected utility representation

**Def**: A person’s preferences over the lotteries in $\Delta(X)$ (where $X$ is a set of outcomes) can be represented by an *expected utility function* if there exists a function $u : X \rightarrow \mathbb{R}$ such that for any $p, q \in \Delta(X)$,

$$p \succ q \iff EU(p) \equiv \sum_{i=1}^{n} p_i u(x_i) > \sum_{i=1}^{n} q_i u(x_i) \equiv EU(q).$$

**Axioms that imply EU-representable preferences**

Consider a set of outcomes $X$, and lotteries $\Delta(X)$.

A1 (completeness): For any $p, q \in \Delta(X)$, if $p \succ q$ then not $q \succ p$.

A2 (neg. transitivity): For any $p, q, r \in \Delta(X)$, if $p \succ q$, then either $p \succ r$ or $r \succ q$. (Recall that neg trans implies trans.)

A3 (“Archimedian”, or continuity): For $p, q, r \in \Delta(X)$ s.t. that $p \succ q \succ r$, $\exists \alpha, \beta \in (0, 1)$ s.t. $(\alpha p, (1 - \alpha)r) \succ q \succ (\beta p, (1 - \beta)r)$.

A4 (substitution, or “independence”): For any $p, q, r \in \Delta(X)$, if $p \succ q$, then for $\alpha \in (0, 1)$, $(\alpha p, (1 - \alpha)r) \succ (\alpha q + (1 - \alpha)r)$.

**Th**: A dmkr prefers $\succ$ on $\Delta(X)$ can be represented by an expected utility function iff they satisfy A1, A2, A3, and A4.

**Compound lotteries**

Objects like $(\alpha p, (1 - \alpha)r)$ are *compound lotteries* – the lottery you get when you have an $\alpha$ chance of getting the lottery $p$, and a $1 - \alpha$ chance of getting the lottery $r$.

**The Allais paradox, an example of a common violation of A4**

You have a choose between two gambles:

$G_1 = \$1m$, $G_2 = (.01 \text{ on } \$0, .89 \text{ on } \$1m, .10 \text{ on } \$5m)$,
and then between two more,

\[ G_3 = (.89 \text{ on } 0, .11 \text{ on } 1m), \quad G_4 = (.9 \text{ on } 0, .1 \text{ on } 5m) \].

People often express the preferences \( G_1 \succ G_2 \) and \( G_4 \succ G_3 \). But this violates the substitution axiom, as show by

\[
\begin{array}{ccc}
\text{probability} & \text{probability} \\
\hline
.01 & .10 & .89 & \text{vs.} & .01 & .10 & .89 \\
G_1 & 1m & 1m & 1m & G_3 & 1m & 1m & 0m \\
G_2 & 0m & 5m & 1m & G_4 & 0m & 5m & 0m \\
\end{array}
\]

Now we show that you can’t represent these expressed preferences with an EU function (which we already know must be the case from the Theorem above):

Let \( X = \{0, 1m, 5m\} \). Let \( u(5m) = 1, \ u(0) = 0, \) and keep \( u(1m) \) “free.” (Why can we do this without losing any generality?)

\( G_1 \succ G_2 \) implies \( u(1m) > .01 \cdot 0 + .89 \cdot u(1m) + .10 \cdot 1 \), but this implies

\[ u(1m) > .89u(1m) + .1 \]

or

\[ .11u(1m) > .1 \]

And \( G_3 \prec G_4 \) implies

\[ .89u(0) + .11u(1m) < .9u(0) + .1u(5m) \]

\[ .11u(1m) < .1 \]

But this is a contradiction, so there is no eu function that can represent these preferences.

Utility functions over continuous sets of outcomes and risk attitudes
**Definition:** A person is risk averse if she strictly prefers to receive the expected value of a lottery for sure to the lottery itself. Formally, if for numerical outcomes \( x \) and \( y \), she has
\[
\alpha x + (1 - \alpha)y \succ (\alpha x, (1 - \alpha)y)
\]
for all \( \alpha \in (0, 1) \).

**Two concepts of risk attitude**

1) Absolute risk attitude: The idea just presented, curvature of a utility function defined on a continuous, numerical set of outcomes.

2) Relative risk attitude: “Rumsfeld is more risk acceptant than Powell.” Here, idea is that A is more risk acceptant (averse) than B if A will take (reject) gambles that B would reject (accept). This does not presume any underlying metric in outcomes.

**Definition of a normal form game**

A normal form game is

1. A set of players \( I = \{1, 2, \ldots, n\} \).
2. A set of strategies available to each player \( i \), \( S_i \).
3. A utility function \( u_i : S_1 \times S_2 \times \ldots \times S_n \to \mathbb{R} \) for each player \( i \).

Formally, we will often write \( \Gamma = \langle I, S_i, u_i \rangle \).

**Payoff notation**

In general, \( u_i(s_i, s_j) \) is the utility number for player 1 assigned when 1 chooses strategy \( s_i \in S_1 \) and player 2 chooses \( s_j \in S_2 \).

What if the number of players is more than 2? Then \( u_i(s_1, s_2, s_3, \ldots) \) is the utility payoff player 1 gets when she chooses \( s_1 \) and player 2 chooses \( s_2 \in S_2 \), etc. For example, \( u_1(F, F, F, P, F, P) \) means what in a six player SoN game?

For convenience, we will often write (for \( n > 2 \) player games) \( u_i(s_i, s_{-i}) \) for player \( i \)'s payoff, where \( s_i \) is the strategy \( i \) is choosing and \( s_{-i} \) is the list of strategies that everyone else is choosing. \((-i \) here means something like “not person \( i \”). Thus in the example in the last paragraph, \( s_i = F \) and \( s_{-i} = (F, F, P, F, P) \).

**Payoffs given beliefs about what other player may choose**

We can extend the notation for utility payoffs for outcomes to represent a player’s expected utility for choosing a given strategy \( s_i \) given that the player has some belief about what the other is likely to do.
Consider a two player game in which 1’s strategies are $S = \{s_1, s_2, s_3, \ldots, s_m\}$ and 2’s strategies are $T = \{t_1, t_2, \ldots, t_n\}$.

Then 1’s belief about player 2’s likely play is a probability distribution on $T$, thus an element $\tau \in \Delta(T)$. Written out, $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$, where $\tau_i = Pr(2 \text{ plays } t_i)$.

For the general case where 1 plays some strategy $s_i \in S$,

$$u_1(s_i, \tau) = \sum_{j=1}^{n} \tau_j u_1(s_i, t_j).$$

**Definition of a best reply**

**Defn:** $s_i$ is a best reply given belief $\tau$ for player 1 if she has $u_1(s_i, \tau) \geq u_1(s_j, \tau)$ for all strategies $s_j \in S_1$.

**Defn:** The set of best replies for player 1 is $BR_1(\tau) = \{s_i : s_i \text{ is a best reply given } \tau\}$.

(Note that the “:” in the last definition reads “such that.”)

**Never a best reply**

First concept of a bad strategy that a rational player could rule out:

**Defn:** $s_i \in S$ is never a best reply if there does not exist $\tau \in \Delta(T)$ s.t. $u(s_i, \tau) \geq u(s, \tau)$ for all $s \in S$.

(Note: Now we are talking about two player games and I am using $S$ for 1’s strategy set and $T$ for 2’s strategy set.)

Question: Is it true in general that if a strategy $s_i$ gives better payoffs than another strategy $s_j$ for every possible strategy choice by player 2, then it is does better for any belief $\tau \in \Delta(T)$? Yes.

Claim: For two strategies (acts) $s_i$ and $s_j$, $u(s_i, \tau) > u(s_j, \tau)$ for all $\tau \in \Delta(T)$ iff $u(s_i, t) > u(s_j, t)$ for all $t \in T$.

**The idea of mixed strategies**

Suppose we give players the option of choosing a probability distribution on the set of available pure strategies.

If 1’s set of pure strategies (actions, acts) is $S = \{s_1, s_2, \ldots, s_m\}$, then $\Delta(S)$ is the set of
mixed strategies.

This is exactly parallel to the idea of \( \tau = (\tau_1, \tau_2, \ldots) \), except there we were talking about \( \tau \) as player 1’s beliefs about 2’s likely play, and here we are talking about \( \sigma \) as 1’s actual strategy.

If player 1 has \( m \) available strategies in an \( m \times n \) normal form game, we write

\[ \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m), \]

where \( \sigma_i \) is the probability that player 1 choose strategy \( s_i \in S \).

**Expected payoff given a mixed strategy \( \sigma \) and a belief \( \tau \)**

Defn: A person’s eu for the mixed strategy \( \sigma \in \Delta(S) \), given beliefs about the other’s play \( \tau \in \Delta(T) \), is

\[
u(\sigma, \tau) = \sum_{i=1}^{m} \sigma_i u(s_i, \tau) = \sum_{i=1}^{m} \sigma_i \sum_{j=1}^{n} \tau_j u(s_i, t_j).
\]

**Strongly dominated strategies**

A second concept of what a “bad strategy” would be for a rational player in a game situation:

Defn: \( \sigma \in \Delta(S) \) is strongly dominated by \( \sigma' \in \Delta(S) \) if \( u(\sigma', t) > u(\sigma, t) \) for all \( t \in T \).

In words, strategy \( \sigma \) gives a worse expected payoff for player 1 than does strategy \( \sigma' \) no matter what player 2 chooses.

**Equivalence of the two concepts in 2 person games**

**Thm:** If there are 2 players, then \( \sigma \in \Delta(S) \) is never a best reply iff there exists \( \sigma' \in \Delta(S) \) s.t. \( \sigma \) is strongly dominated by \( \sigma' \).

In words, if there is no set of beliefs for which a strategy is optimal (a best reply), then there must exist some other strategy that is better than it in all possible circumstances regarding the other player’s play.