Lecture 2: Ultrasound as a Wave
Lecture Objectives

• What is a wave?

• Derive the acoustic wave equation

• Describe an acoustic wave using standard terminology

• Wave properties
Waves Defined

• A disturbance or oscillation that travels through space and time, accompanied by a transfer of energy from one point to another, with little or no permanent displacement of matter.
Waves Defined

Longitudinal Wave

Animation courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State
Waves Defined

Shear Wave

Animation courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State
Waves Defined
Surface Wave

Animation courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State
Waves Defined

We will focus on longitudinal waves in this course

Animation courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State
Mathematical Representation

• The wave equation:

$$\nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}$$
Derivation of the acoustic wave equation

1. **Conservation of momentum**: difference in pressure between two points in space (pressure gradient) gives rise to a force between those two points:
   \[ \text{force} = \text{mass} \cdot \text{acceleration} \propto -\text{pressure gradient} \]

Assume an infinitesimal mass:
\[ m = \rho A dx \]

Velocity of the particle along the \( x \) direction:
\[ v = \frac{dx}{dt} \]

\[ dF = -dPA \rightarrow \rho A dx \frac{dv}{dt} = -dPA \rightarrow \rho \frac{dv}{dt} = -\frac{dP}{dx} \]

And since both velocity and pressure can have space and time dependencies, it is more accurate to write the relationship as partial derivatives:
\[ \rho \frac{dv}{dt} = -\frac{\partial P}{\partial x} \]
Derivation of the acoustic wave equation

2. Conservation of energy: if velocity at \( x + \Delta x \) is greater than it is at \( x \), the pressure at point \( x + \Delta x \) will be less than the pressure at point \( x \). This pressure drop is related to compressibility as follows:

\[
\frac{-\partial P}{\partial t} = K \frac{\partial v}{\partial x}
\]

Take equation 1, divide both sides by \( \rho \) and take the derivative with respect to \( x \):

\[
\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial x} \Rightarrow \frac{\partial}{\partial x} \frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial^2 P}{\partial x^2} \Rightarrow \frac{\partial}{\partial t} \frac{\partial v}{\partial x} = -\frac{1}{\rho} \frac{\partial^2 P}{\partial x^2}
\]

Substituting \(-\frac{1}{K} \frac{\partial P}{\partial t}\) for \( \frac{\partial v}{\partial x} \), we have:

\[
2. \quad \frac{\partial^2 P}{\partial t^2} = \frac{K}{\rho} \frac{\partial^2 P}{\partial x^2}
\]
Derivation of the acoustic wave equation

Compressibility and the speed of sound are related as: 

\[ c = \sqrt{\frac{K}{\rho}} \]

Therefore, equation 2. in one dimension becomes:

\[ \frac{\partial^2 P}{\partial t^2} = c^2 \frac{\partial^2 P}{\partial x^2} \]

In three dimensions:

\[ \frac{\partial^2 P}{\partial t^2} = c^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) P \]

And since \( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \nabla = \nabla^2 \), the most general form of the acoustic wave equation is:

\[ \frac{\partial^2}{\partial t^2} P(x, y, z, t) = c^2 \nabla^2 P(x, y, z, t) \]
Mathematical Representation

• The wave equation:

\[ \nabla^2 P(x, y, z, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} P(x, y, z, t) \]

• Many phenomena can be represented by this equation!
Mathematical Representation

• The wave equation:

\[ \nabla^2 P(x, y, z, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} P(x, y, z, t) \]

• Many phenomena can be represented by this equation!
  – “The most incomprehensible thing about the universe is that it is comprehensible.” - Albert Einstein
Solutions to the Wave Equation

• The most common solution to this equation is:

\[ A e^{i\omega t + i\vec{k} \cdot \vec{r}} = A e^{i\omega t + i(k_x x + k_y y + k_z z)} \]

• For this lecture, we will assume propagation in the z-direction and the solution simplifies to:

\[ A e^{i\omega t + i k z} \]
Some Terminology

\[ Ae^{i\omega t + i\vec{k} \cdot \vec{r}} \]

- \( \omega \): radial frequency
- \( f \): frequency = \( \omega / 2\pi \)
- \( T \): period = 1/f
- \( A \): amplitude
- \( \vec{k} \): wave vector
  - \( |\vec{k}| = \omega / c = 2\pi / \lambda \)
  - \( \hat{k} \) points in direction of propagation
- \( \lambda \): wavelength = \( c / f \)
Phase, $\varphi$

- Phase describes how many cycles a signal has accumulated.

- It is *always* relative.

- $k$ describes how quickly phase accumulates in space ($\varphi = kz$).

- $\omega$ describes how quickly phase accumulates in time ($\varphi = \omega t$).
Wavefronts

• A surface of equal phase

• What shape are the wavefronts of:
  \[ A \ e^{i\omega t + ikz} \]
Wavefronts

• A surface of equal phase

• What shape are the wavefronts of:

\[ A \ e^{i\omega t + ikz} \]
Wavefronts

• A surface of equal phase

• What shape are the wavefronts resulting from a point source on: $A e^{i\omega t + ikz}$
Wavefronts

• A surface of equal phase

• What shape are the wavefronts resulting from a point source on: $A e^{i\omega t + ikz}$
Continuous wave vs. Pulsed Wave

Pulse Duration = Period x number cycles in the pulse

Period (time for one cycle)
Components of a multi-pulse signal

- **Pulse length** (units: seconds)
- **Repetition Period**, or more often denoted: 
  
  Pulse repetition frequency (PRF) = \(1/\text{RP}\)

\[
\text{Duty Cycle} = \frac{\text{PL}}{1/\text{PRF}} = \frac{\text{PL} \times \text{PRF}}{\text{RP}}
\]

These three properties of a pulse will be revisited when discussing resolution, driving transducers, and therapeutic effects.
Wave Phenomena

• **Superposition** - The net displacement of the medium at any point in space or time, is simply the sum of the individual wave displacements.
Wave Phenomena

- **Interference** is the result of superposition when two waves are in the same place at the same time.
- The result can be **constructive** or **destructive** interference.
Wave Phenomena

- **Diffraction** - The apparent bending of waves around small obstacles and the spreading out of waves past small openings.
Wave Phenomena

- **Reflection** - a fraction of the wave “bounces back” at a boundary

- **Refraction** - change in direction of a wave due to a change in its medium.
Standing Waves

Interference of two identical harmonic waves traveling in opposite directions
Questions

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• Office hours: TBD