Class 6 - Beam Formation

- Diffraction
- Huygen’s Principle
- Some Math with a Very Cool Result
- Beam Characteristics
- Demo
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Point Source
Diffraction

• Diffraction is a phenomenon that affects all types of waves, and is described as “the deviation of waves from their rectilinear paths that cannot be described as reflection or refraction.

• Typically occurs at the edges or corners of apertures.
Real World Examples
Class 6

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Huygen’s Principle

- Approximate the transducer as a bunch of point sources
- Add them together
- Simulate curved transducer
Near Field and Far Field

D = diameter

Near Field or Fresnel Zone

Focal Distance

Far Field or Fraunhofer Zone

\[ FD = \frac{D^2}{4\lambda} \]
Beam

\[ P(z,t) = 2P_0e^{j(\omega t - kz)} \sin\left(\frac{\pi}{\lambda}\left(\sqrt{z^2 + \left(\frac{D}{2}\right)^2} - z\right)\right) \]

- Last axial maxima is at \( z = D^2/4\lambda \), where \( D = \) transducer diameter
- Max intensity is at minimum width
Near Field often not shown
Beam

Sidelobes

f = 2 MHz
f = 3.5 MHz
f = 5 MHz
a = 0.25 cm
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Huygens-Fresnel Principle

The pressure at some point in space is the sum of all the spherical waves emanating from all points on the aperture.

Where do we think this is going?

\[
P(r_0) = \frac{1}{j\lambda} \int \int_{S} P(r_1) \frac{e^{jkr_{01}}}{r_{01}} \cos \theta dS
\]

\[
I \sim \frac{1}{r^2} \quad P \sim \frac{1}{r}
\]

For convenience, we will use

\[
\sin(kr_{01}) = \frac{1}{j} e^{jkr_{01}}
\]

\[
\cos \theta \quad \text{obliquity term}
\]
Huygens-Fresnel Principle

Assume in far field
\[ |r_{01}| \approx z \]

Assume close to axis
\[ \cos \theta \approx 1 \]
Huygens-Fresnel Principle

\[ P(r_0) = \frac{1}{j\lambda z} \int \int_{S} P(r_1) e^{jkr_{01}} dS \]

\[ P(x_0, y_0) = \frac{1}{j\lambda z} \int \int P(x_1, y_1) e^{jkr_{01}} dS \]

\[ P(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} \int \int P(x_1, y_1) e^{2z\left[(x_0-x_1)^2 + (y_0-y_1)^2\right]} dx_1 dy_1 \]

- First, replace vector length \( |r_{01}| \) with
  \[ |r_{01}| = \sqrt{z^2 + (x_0 - x_1)^2 + (y_0 - y_1)^2} \]

- Factor out \( z \):
  \[ |r_{01}| = z\sqrt{1 + \left(\frac{x_0 - x_1}{z}\right)^2 + \left(\frac{y_0 - y_1}{z}\right)^2} \]

- Use Taylor series for \( |b| \leq 1 \)
  \[ \sqrt{1+b} = 1 + \frac{1}{2} b + \frac{1}{8} b^2 + \cdots \]

- Therefore, for exponent (exponent in equation still sensitive to \( x \) and \( y \)):
  \[ |r_{01}| = z\left[1 + \frac{1}{2} \left(\frac{x_0 - x_1}{z}\right)^2 + \frac{1}{2} \left(\frac{y_0 - y_1}{z}\right)^2\right] \]
Fraunhofer Approximation

Remember,

\[ FD = \frac{D^2}{4\lambda} \]

Let’s look at

\[ z > \frac{D^2}{4\lambda} \]
Fraunhofer Approximation

\[ P(x_0, y_0) = \frac{e^{j kz}}{j \lambda z} \int \int P(x_1, y_1) e^{\frac{jk}{2z} [(x_0-x_1)^2 + (y_0-y_1)^2]} dx_1 dy_1 \]

\[ P(x_0, y_0) = \frac{e^{j kz}}{j \lambda z} e^{\frac{jk}{2z} [x_0^2 + y_0^2]} \int \int P(x_1, y_1) e^{\frac{jk}{2z} [x_1^2 + y_1^2]} e^{-\frac{jk}{z} [x_0 x_1 + y_0 y_1]} dx_1 dy_1 \]

quadratic weighting function \( \sim \) constant

\[ z > \frac{D^2}{4\lambda} \] far field

\[ z > x_1^2 + y_1^2 \] close to axis
Spatial Frequencies

\[ f_x = \frac{1}{\lambda} \frac{x_1}{z} \]

\[ f_y = \frac{1}{\lambda} \frac{y_1}{z} \]
Fraunhofer Approximation

\[ P(x_0, y_0) = \frac{e^{jkz} e^{\frac{jk}{2z}x_0^2}}{j\lambda z} \int \int P(x_1, y_1) e^{-\frac{jk}{z}x_0x_1 + y_0y_1} \, dx_1 \, dy_1 \]

\[ P(x_0, y_0) = \frac{e^{jkz} e^{\frac{jk}{2z}x_0^2}}{j\lambda z} \int \int P(x_1, y_1) e^{-j2\pi x_0f_x + y_0f_y} \, dx_1 \, dy_1 \]

\[ P(x_0, y_0) = \frac{e^{jkz} e^{\frac{jk}{2z}x_0^2}}{j\lambda z} \text{FT} \left\{ \text{Aperture}(x_1, y_1) \right\} \]

\[ f_x = \frac{x_1}{\lambda z} \]

\[ f_y = \frac{y_1}{\lambda z} \]
Fraunhofer Approximation

\[ P(x_0, y_0) \sim FT\{Aperture(x_1, y_1)\} \]
Point Source:

- constant pressure everywhere

Aperture Plane

Field (Observation Plane)

F.T.
Example

Examples

Two Point Sources:

\[
\cos(\pi D f_x)
\]

Aperture Plane  Field (Observation Plane)
Rectangular Aperture:

\[
\text{rect} \quad \rightarrow \quad \text{sinc}
\]

Aperture Plane

Observation Plane
Sidelobes

![Graph showing relative pressure vs distance off-axis with curves for different frequencies (f = 2 MHz, f = 3.5 MHz, f = 5 MHz) and a = 0.25 cm.]
Directivity Pattern

- Analogous to antenna theory for circular aperture (CW)
- First sidelobes occur at ~-17dB

\[ I(\phi) = I(\phi = 0) \left[ \frac{2J_1(ka \sin \phi)}{ka \sin \phi} \right]^2 \]

Eq 3.25 Shung
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Transition Zone

- A higher transducer frequency (shorter wavelength) will result in a longer near field, as will a larger diameter element.
Focal Distance

$$FD = \frac{D^2}{4\lambda}$$

FD longer for __________ diameters
FD longer for __________ frequencies
Focal Width

\[ W = \frac{1.22\lambda}{D} \]

Beam width at the focal distance

W bigger for ___________ diameters

W bigger for ___________ frequencies

Shung p57
Beam Divergence

\[ \sin(\theta) = \frac{1.2 \lambda}{D} \]

Divergence less for ____________ diameters

Divergence less for ____________ frequencies
Transition Zone

- A higher transducer frequency (shorter wavelength) will result in a longer near field, as will a larger diameter element.
* Highly focused transducers have an $f\# \leq 1$
Low f-number transducer

High f-number transducer

Focal plane

\[ f \# = \frac{FD}{D} \]
Focused Transducer Beamplot
Focal Width

Beam width at the focal distance

\[ W = \frac{1.22 \lambda F}{D} \]
Focal Length

\[ l = 3 \left( \frac{FD}{D} \right)^2 \lambda \]

Focal length at the focal distance

Christensen p112
Axial Resolution

Lateral Resolution

Resolvable

Not Resolvable

Resolvable

Not Resolvable

Resolvable

Not Resolvable
Beam Width

Best at focal distance
Worse at depth
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Beam Width

Best at focal distance
Worse at depth
Near Field and Far Field

\[ \text{D} = \text{diameter} \]

\[ F = \frac{D^2}{4\lambda} \]

Near Field or Fresnel Zone

Focal Distance

Far Field or Fraunhofer Zone
Focal Width

\[ W = \frac{1.22 \lambda F}{D'} \]

\[ D' = D \cos \theta \]

\[ \cos \theta \quad \text{obliquity term} \]

Beam width at the focal distance
The Transmitted Axial Field

Back to Huygen-Fresnel, but instead of computing \( p(x, y) \) we will compute \( p(z, t) \) [meaning we only worry about the on-axis pressure field...]

\[
P(\bar{r}_0) = \frac{1}{j\lambda} \iint_{S} P(\bar{r}_1) e^{jk|r_{01}|} \cos \theta \, dS
\]

\( p(z, t) \)

Equals 1 because we are on-axis

Bring back the temporal component

\[
p(z_0, t) = \frac{1}{j\lambda} \iint_{S} p(r_1, t) \frac{e^{j(\omega t - k|r_0|)}}{|r_{01}|} \, dS
\]
The Transmitted Axial Field

\[ p(z_0, t) = \frac{1}{j\lambda} \iint_{\rho, \theta} p(r_1, t) \frac{e^{j(\omega t - k|r_{01}|)}}{|r_{01}|} \rho \, d\rho \, d\theta \]

\[ \rho = \sqrt{x^2 + y^2} \]

\[ \theta = \tan^{-1} \frac{y}{x} \]
The Transmitted Axial Field

For on-axis distances: \( r_{01} = \sqrt{\rho^2 + z^2} \)

Assume: \( p(r_1, t) = p_0 \)

\[
p(z, t) = \frac{1}{j\lambda} \int_0^a p_0 \frac{e^{j(\omega t - k\sqrt{\rho^2 + z^2})}}{\sqrt{\rho^2 + z^2}} \rho d\rho \int_0^{2\pi} d\theta
\]

\[
p(z, t) = 2p_0 e^{j(\omega t - kz)} \sin \frac{k\sqrt{z^2 + a^2 - z^2}}{2}
\]