Homework 1 – Due Thursday Oct 2 – Covers Classes 1-3

1. (8 pts) The wave function describing a wave travelling on a string is given by:

\[ y = 0.1 \sin \left( \frac{20}{17} \pi x - 200 \pi t \right) \]

where \( y \) is the displacement in [mm], \( t \) is time in [sec] and \( x \) is the distance from the origin in [m].

(a) Find the frequency of the wave in Hz
(b) Find the wavelength
(c) What is the phase difference between the wave at \( x=0.25 \) and \( x=1.1 \)?
(d) Write down the wave equation of a wave travelling in the opposite direction with double amplitude and frequency

(a) frequency of the wave: \( f = \frac{w}{2\pi}; w=200\pi \gg f=100\text{Hz} \)
(b) wavelength: \( \lambda = \frac{2\pi}{k}; k=\frac{20\pi}{17} \gg \lambda = 1.7\text{m} \)
(c) phase difference between the wave at \( x=0.25 \) and \( x=1.1 \): \( 1.1-0.25 = 0.85 \) which is exactly half the wavelength. So the phase difference is 180 degrees, or \( \pi \).
(d) wave equation of a wave travelling in the opposite direction with double amplitude and frequency:

The wave equation is as follows:

\[ y = 0.2 \sin \left( -\frac{40}{17} \pi x - 400 \pi t \right) \]

note that since the medium hasn’t changed, the wave velocity remains the same, which means that since we doubled the frequency we have to reduce the wavelength in half.

2. (8 pts) Calculate the reflection coefficient \( R \) and the ratio of reflected power to incident power for each of the following interfaces (Assuming normal incidence)

a) Fat/Bone
b) Muscle/Liver
c) Fat/Lung

\[ \begin{align*}
a) & \quad R = -0.62 \quad R^2 = 0.38 \\
b) & \quad R = 0.003 \quad R^2 = 0.000009 \\
c) & \quad R = -1 \quad R^2 = 1
\end{align*} \]

3. (8 pts) Consider the ultrasound imaging configuration shown on the right:

If at least 20 \( \mu \text{W/cm}^2 \) must be returned to the transducer from the reflection at the muscle/fat interface for a good signal-to-noise ratio, calculate the power density that must be transmitted by the transducer at a frequency of \( f = 2 \text{ MHz} \). Neglect any losses due to transverse spreading of the beam.
Numbers can vary, depending on the tissue property values you assume:
\[ R = \frac{(1.65 - 1.33)}{(1.65 + 1.33)} = 0.107 \]
\[ R^2 = 0.01 \rightarrow 20 \text{dB} \]

Attenuation loss = 1.2 dB/cm/MHz * 12 cm * 2 MHz = 28.8 dB

\[ 48.8 \text{dB} = 10 \log (I/I_0) \]
\[ I_0 = I/10^{-4.88} = 20 \times 10^{-6} \text{ W/cm}^2 \times 10^{-4.88} \]
\[ I_0 = 1.5 \text{ W/cm}^2 \]

OR

\[ a = 1.2/8.668 = 0.138 \]
\[ I_4 = I_1 e^{-2a f L^2} R^2 \]
\[ 20 \times 10^{-6} \text{ W/cm}^2 = I_1 e^{(-2 \times 0.138 \times 12^2) \times 0.01} = I_1 \times 0.000013 \]
\[ 1.5 \text{ W/cm}^2 \]

4. (4 pts) A wave incident from the left of the boundary shown below has a pressure reflection coefficient given by \( R_L \) and a power reflection coefficient given by \( R_L^2 \),

\[ Z_1 \]

\[ R_L \rightarrow \]

\[ Z_2 \]

\[ R_R \]

Find the pressure reflection coefficient \( R_R \) and the power reflection coefficient for a wave incident on the same boundary from the right. Put both in terms of \( R_L \).

\[ R_R = -R_L, \quad R_R^2 = R_L^2 \]

5. (9 pts) Here we have a simple model of two ultrasound beams (originating at Tx1 and Tx2) crossing the skull to a point at the center of the skull. The skull has a speed of sound different from the speed of sound in and outside the skull. The skull will aberrate the beam (the two beams will have different phases).
Assume the transducers have a center frequency of 1 MHz.

a. Relative to Tx1, what phase should be applied to Tx2 in order to focus at the center of the circle? Use the following values:
   i. \( a = 2 \text{ mm} \)
   ii. \( r = 2 \text{ cm} \)
   iii. \( d = 2 \text{ cm} \)
   iv. \( c_0 = 1540 \text{ m/s} \)
   v. \( c_1 = 1900 \text{ m/s} \)

b. What complicating factors would occur if the path between the transducer and the target didn’t intersect the circle at a right angle?

c. Assuming the circle was much more complicated with a variety of thicknesses and sound speeds, can you think of an experimental way to recover the phase that should be applied?

\[
(\text{a}) \quad d\phi = (k_b^*a + k_s^*a) - 2k_b^*a
\]

where \( k_b \) is the wave number in brain and \( k_s \) is the wave number in skull

\[
= a*w/c_b + a*w/c_s - 2*a*w/c_b = 1.55 \text{ radians}
\]

OR you can do it in terms of delays

\[
\frac{a}{c_b} + \frac{a}{c_s} - 2\frac{a}{c_b} = 2.46e-7
\]

(b) Refraction will change the beam path length

(c) Anything that works is acceptable. Ex: put a hydrophone inside and measure delays, put a source inside and listen at element locations to measure delays, use a CT/MR scan to estimate properties.
6. (4 pts) Assume you want to measure the speed of sound of a rectangular object of thickness, \( h \). You immerse the object in 20°C water and measure the time it takes for a pulse to arrive at the receiver. You then repeat the experiment without the object.

   a. What is the speed of sound in the object? Use the following values:
      i. \( h = 1 \text{ mm} \)
      ii. arrival time with object present is 26 \( \mu \text{s} \)
      iii. arrival time without object present is 24 \( \mu \text{s} \)

   b. How does your answer change if you assume the temperature of the water is 37°C?

   (a) \( c_1 = \frac{h}{(dt+h/c_0)} = 374 \text{ m/s} \) where \( c_0 \) is the speed of sound in water at 20 C (1480)
   (b) Change \( c_0 \) for water at 37 (1510 m/s) = 376 m/s This is a bit ambiguous since we didn't clarify how the water or object speed of sound changed with temperature. Sorry!

7. (4 pts) Relate the bulk modulus, density, and speed of sound in both water and fat. Why is the speed of sound lower in fat (is it because of the bulk modulus or the density)? What is it about that parameter that varies significantly between fat and water?

   Fat: \( B = 1.9 \text{ Mpa}, \rho = 928 \text{ kg/m}^3 \); water: \( B = 2.66 \text{ MPa}, \rho = 1000 \text{ kg/m}^3 \)

   The bulk modulus drives the difference since the density would actually suggest that the speed of sound should be higher in fat. Recall \( c = \sqrt{B/\rho} \)