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Problem Set #1

Rad 226a/BioE 326a

1. **Deriving the Bloch Equations.**

a) Starting with

$$\frac{d\vec{M}(t)}{dt} = \vec{M}(t) \times \gamma \vec{B}(t), \qquad \vec{B}(t) = B_{1x}\vec{x} + B_{1y}\vec{y} + B_0\vec{z}$$

Derive the Bloch equations in the laboratory frame without relaxation. Namely:

$$\frac{dM_x(t)}{dt} = \gamma\left[M_y(t)B_0 - M_z(t)B_{1y}\right]$$
$$\frac{dM_y(t)}{dt} = \gamma\left[M_z(t)B_{1x} - M_x(t)B_0\right]$$
$$\frac{dM_z(t)}{dt} = \gamma\left[M_x(t)B_{1y} - M_y(t)B_{1x}\right]$$

b) Starting from the full Bloch equations in the laboratory frame:

$$\frac{dM_x(t)}{dt} = \gamma\left[M_y(t)B_0 - M_z(t)B_{1y}\right] - \frac{M_x(t)}{T_2}$$
$$\frac{dM_y(t)}{dt} = \gamma\left[M_z(t)B_{1x} - M_x(t)B_0\right] - \frac{M_y(t)}{T_2}$$
$$\frac{dM_z(t)}{dt} = \gamma\left[M_x(t)B_{1y} - M_y(t)B_{1x}\right] - \frac{M_z(t) - M_0}{T_1}$$

Derive the Bloch equations for a frame of reference rotating at a frequency ω clockwise about the z axis (left handed rotation).

c) Show that the following are solutions to the Bloch equations in the rotating frame, assuming initial magnetization in the transverse plane and free induction decay (i.e.: $\vec{B}(t) = B_0\vec{z}$):

$$M_x(t) = M_0 \cos\left[(\gamma B_0 - \omega)t + \phi\right]e^{-t/T_2}$$
$$M_y(t) = -M_0 \sin\left[(\gamma B_0 - \omega)t + \phi\right]e^{-t/T_2}$$
$$M_z(t) = M_0\left(1 - e^{-t/T_1}\right)$$

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Problem Set #1 (cont.)

Rad 226a/BioE 326a

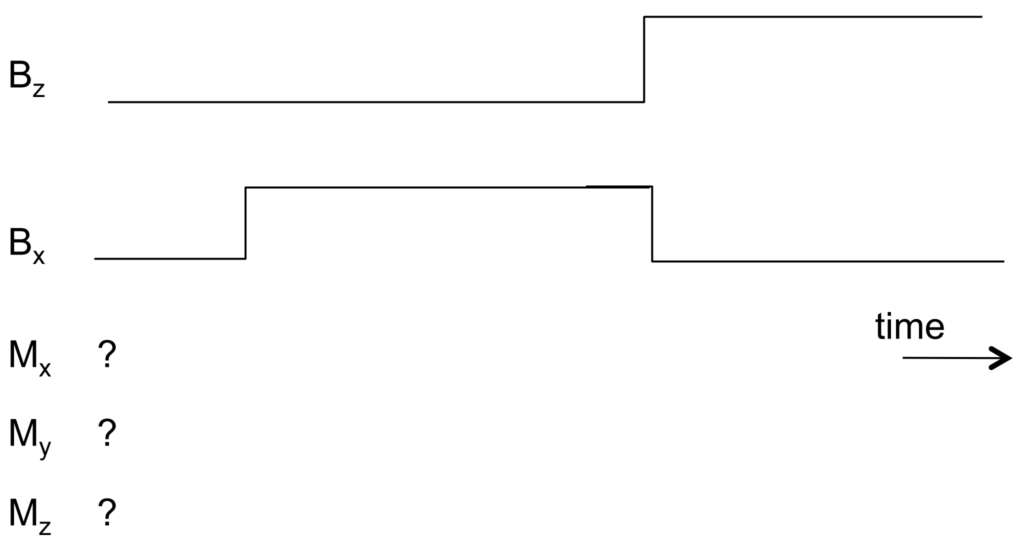
2. **Nuclear Induction.** This question concerns the paper by F. Bloch, "Nuclear Induction", Physical Review, 76:460-474, 1946.

a) Verify equation 16. What is the physical interpretation for this value of θ?

b) Which of the following concepts are discussed in Bloch's 1946 paper?

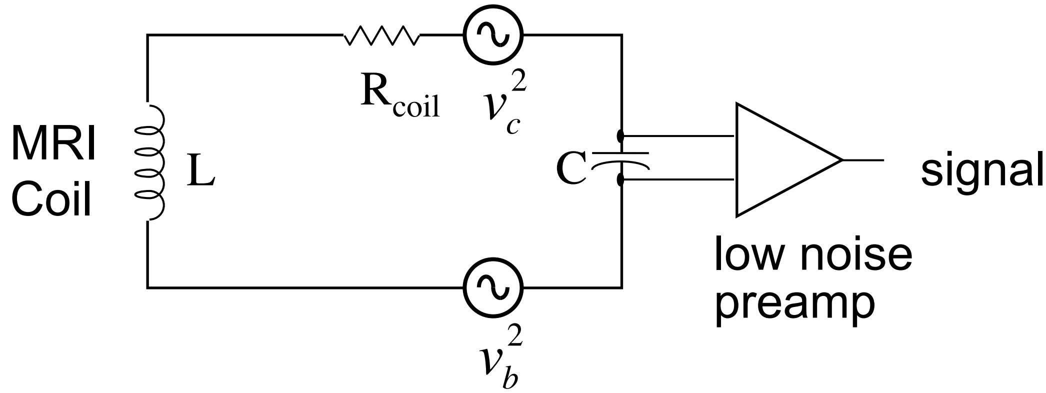
1. Continuous wave (CW) NMR
2. Pulsed NMR
3. Prepolarized NMR
4. Quadrature excitation
5. Adiabatic excitation
6. T_1
7. T_2
8. Doping samples with small amounts of paramagnetic material to reduce T_1
9. Imaging

3. **Block Equations**. A sample of water is left to reach thermal equilibrium in zero magnetic field. A magnetic field is then turned on in the x-direction. After some time, the magnetic field is switched from the x-direction to the z-direction. Describe the behavior of the net ^1H magnetic moment from the sample over time. Can you describe the behavior of an individual nuclear spin? Can you explain T_1 and T_2?



Problem X. MRI Sensitivity and SNR

Consider the following simplified model of a receive coil and preamp,



where v_c and v_b represent coil and body noise sources respectively and···

$M_0 = \rho \dfrac{\gamma^2 \hbar^2 B_0}{4kT}$ = steady-state longitudinal magnetization,

ρ = spins/unit volume,

$\hbar$ = Planck's constant/2π = 1.05×10^{-34} Joule·s,

k = Boltzmann's constant = 1.38×10^{-23} J/K,

T = absolute temperature.

Johnson noise spectral density is given by $v_n^2 = 4kTR$,

$R_{coil} \propto \sqrt{\omega}$ (due to RF skin depth effects),

$R_{body} \propto \omega^2$ (from inductive losses, ignoring dialectric losses),

Signal (voltage induced in coil) $\propto \dfrac{dM}{dt}$.

PhD graduate student A, who works in an imaging lab, claims that the signal-to-noise ratio (SNR) for MRI is linearly proportional to B_0. Whereas, student B, who works in a chemistry lab studying protein structure, claims that the SNR for NMR goes as $B_0^{7/4}$. Derive an expression for SNR and determine who is correct.