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Problem Set #2

Rad 226

1. **Invariance of the trace**.
   Prove that the trace of the matrix representation of an operator is independent on the corresponding basis.

2. **Hermitian Operators**.
   a) Prove that the eigenvalues of a Hermitian operator are real.

   b) Prove that the eigenkets corresponding to non-degenerate eigenvalues of a Hermitian operator are orthogonal.

3. Dirac notation.

Using Dirac notation, let $\left|f\right\rangle$ and $\left|g\right\rangle$ be kets in a Hilbert space and $\hat{A}$ an operator in that space. For each of the following expressions, state whether the result is a ket, bra, operator, or scalar.

a. $\langle f|g\rangle$

b. $\langle f|\hat{A}$

c. $|f\rangle\langle g|$

d. $A|f\rangle\langle g|$

e. $\langle f|\hat{A}^{\dagger}$

f. $\langle f|\hat{A}|g\rangle$

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4. **Projection Operators**

Let $|\alpha\rangle$ and $|\beta\rangle$ form a complete orthonormal basis set for a two-dimensional system. Consider the wavefunction

$$|\varphi\rangle = \lambda_1|\alpha\rangle + \lambda_2|\beta\rangle \quad \text{where} \quad |\lambda_1|^2 + |\lambda_2|^2 = 1$$

and its corresponding projection operator $\hat{P}_\varphi = |\varphi\rangle\langle\varphi|$.
The information content of the system can be measured by the quantity $\mathrm{Tr}\left(\hat{P}_\varphi^2\right)$.

a. Case 1. Let λ_2=0 (or equivalently λ_1=0)
This is known as a *pure* state. Namely $|\varphi_1\rangle = |\alpha\rangle$.
Find the eigenkets, eigenvalues, and trace of $\hat{P}_{\varphi_1}$.
Find $\mathrm{Tr}\left(\hat{P}_\varphi^2\right)$.

b. Case 2. Let $\lambda_2 \neq 0$ and $\lambda_2 \neq 0$.
Namely $|\varphi_2\rangle = \lambda_1|\alpha\rangle + \lambda_2|\beta\rangle$.
Find $\mathrm{Tr}\left(\hat{P}_\varphi^2\right)$.

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5. **Superoperators.**
Given operators $\hat{A}$, $\hat{B}$ and $\hat{C}$, prove the following:

a) If $\hat{\hat{A}}\hat{B} = 0$, then $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}$.

b) $e^{\hat{\hat{A}}}\hat{B} = e^{\hat{A}}\hat{B}e^{-\hat{A}}$.

c) If $\hat{\hat{A}}\hat{B} = 0$, then $e^{\hat{\hat{A}}+\hat{\hat{B}}}\hat{C} = e^{\hat{\hat{A}}}e^{\hat{\hat{B}}}\hat{C} = e^{\hat{\hat{B}}}e^{\hat{\hat{A}}}\hat{C}$.