Problem Set #2
Rad 226

1. **Invariance of the trace.**
   Prove that the trace of the matrix representation of an operator is independent on the corresponding basis.

2. **Hermitian Operators.**
   a) Prove that the eigenvalues of a Hermitian operator are real.
   b) Prove that the eigenkets corresponding to non-degenerate eigenvalues of a Hermitian operator are orthogonal.
3. Dirac notation.

Using Dirac notation, let \( |f\rangle \) and \( |g\rangle \) be kets in a Hilbert space and \( \hat{A} \) an operator in that space. For each of the following expressions, state whether the result is a ket, bra, operator, or scalar.

a. \( \langle f | g \rangle \)

b. \( \langle f | \hat{A} \rangle \)

c. \( |f\rangle \langle g| \)

d. \( A |f\rangle \langle g| \)

e. \( \langle f | \hat{A}^\dagger \rangle \)

f. \( \langle f | \hat{A} | g \rangle \)
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Rad 226

4. Projection Operators

Let $|\alpha\rangle$ and $|\beta\rangle$ form a complete orthonormal basis set for a two-dimensional system. Consider the wavefunction

$$|\varphi\rangle = \lambda_1 |\alpha\rangle + \lambda_2 |\beta\rangle$$

where $|\lambda_1|^2 + |\lambda_2|^2 = 1$

and its corresponding projection operator $\hat{P}_\varphi = |\varphi\rangle\langle\varphi|$. The information content of the system can be measured by the quantity $Tr(\hat{P}_\varphi^2)$.

a. Case 1. Let $\lambda_2=0$ (or equivalently $\lambda_1=0$)

This is known as a pure state. Namely $|\varphi_1\rangle = |\alpha\rangle$.

Find the eigenkets, eigenvalues, and trace of $\hat{P}_{\varphi_1}$.

Find $Tr(\hat{P}_{\varphi_1}^2)$.

b. Case 2. Let $\lambda_2\neq0$ and $\lambda_2\neq0$.

Namely $|\varphi_2\rangle = \lambda_1 |\alpha\rangle + \lambda_2 |\beta\rangle$.

Find $Tr(\hat{P}_{\varphi_2}^2)$. 
5. Superoperators.

Given operators \( \hat{A} \), \( \hat{B} \) and \( \hat{C} \), prove the following:

a) If \( \hat{A}\hat{B} = 0 \), then \( e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} \).

b) \( e^{\hat{A}} \hat{B} = e^{\hat{A}} \hat{B} e^{-\hat{A}} \).

c) If \( \hat{A}\hat{B} = 0 \), then \( e^{\hat{A} + \hat{B}} \hat{C} = e^{\hat{A}} e^{\hat{B}} \hat{C} = e^{\hat{B}} e^{\hat{A}} \hat{C} \).