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Problem Set #3
Rad 226

1. The Pauli matrices

Let $\left|\varphi_{x_i}\right\rangle$, $\left|\varphi_{y_i}\right\rangle$, and $\left|\varphi_{z_i}\right\rangle$ for $i = 1,2$ be the eigenkets for $\hat{I}_x$, $\hat{I}_y$, and $\hat{I}_z$.

The Pauli matrices, expressed in the $\left|\varphi_{z_i}\right\rangle$ basis, are given by:

$$\underline{I}_x = \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \underline{I}_y = \tfrac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \underline{I}_z = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

a) Find the corresponding eigenvectors and eigenvalues for $\underline{I}_x$, $\underline{I}_y$, and $\underline{I}_z$.

b) Show by direct matrix calculation that:

$$\sum_{i=1}^{2}\left|\varphi_{x_i}\right\rangle\left\langle\varphi_{x_i}\right| = \sum_{i=1}^{2}\left|\varphi_{y_i}\right\rangle\left\langle\varphi_{y_i}\right| = \sum_{i=1}^{2}\left|\varphi_{z_i}\right\rangle\left\langle\varphi_{z_i}\right| = \hat{E} \text{ (identity operator)}$$

c) Find $\left[\hat{I}_z, \hat{I}_x\right]$, $\left[\hat{I}_z, \hat{I}_y\right]$, and $\left[\hat{I}_x, \hat{I}_y\right]$.

d) Express $\hat{I}_x$, $\hat{I}_y$, and $\hat{I}_z$ in the $\left|\varphi_{y_i}\right\rangle$ basis.

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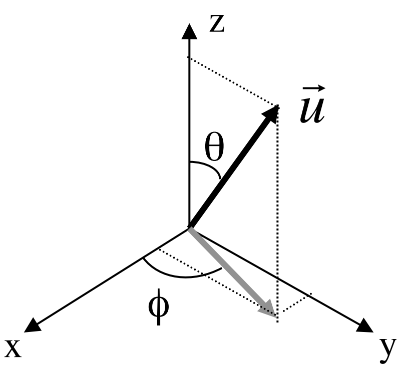
2. **Stern-Gerlach Experiment**

The *x*, *y*, and *z* components of the intrinsic angular momemtum (spin) of a spin-1/2 particle are given by the cyclically commuting operators:

$$\hat{I}_x,\ \hat{I}_y,\ \text{and}\ \hat{I}_z,\ \text{each with eigenvalues} \pm 1/2$$

Consider an arbitrary unit vector $\vec{u}$.



a) What are the eigenvalues of $\hat{I}_u$, the operator corresponding to the spin in the $\vec{u}$ direction?

b) What is the matrix form of $\hat{I}_u$ expressed in the $\{|+\rangle, |-\rangle\}$ basis, where $|+\rangle$ and $|-\rangle$ are the eigenkets of $\hat{I}_z$ corresponding to eigenvalues +1/2 and -1/2 respectively?

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c) The general form of a ket for single spin-1/2 particle is

$$\left|\psi\right\rangle = \alpha\left|+\right\rangle + \beta\left|-\right\rangle \text{ where } \left|\alpha\right|^2 + \left|\beta\right|^2 = 1.$$

Can you find α and β such that $\left|\psi\right\rangle$ represents an unpolarized spin, i.e.

$$\left\langle \hat{I}_x \right\rangle = \left\langle \hat{I}_y \right\rangle = \left\langle \hat{I}_z \right\rangle = 0?$$

d) Consider a collection of spin-1/2 nuclei each described by the same ket

$$\left|s_p\right\rangle = \tfrac{1}{\sqrt{3}}\left(\left|s_x\right\rangle + \left|s_y\right\rangle\right)$$

where $\left|s_x\right\rangle$ and $\left|s_y\right\rangle$ are spin states oriented along the $+x$ and $+y$ directions respectively. This system is said to be in a "pure" state.

Show $\left|s_p\right\rangle$ is normalized.

Find $\left\langle \hat{M}_x \right\rangle$, $\left\langle \hat{M}_y \right\rangle$, and $\left\langle \hat{M}_z \right\rangle$.

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e) Silver atoms leaving the furnace in the Stern-Gerlach experiment can be polarized in any direction. It can be shown that the state of a spin polarized in an arbitrary direction $\vec{u}$ is given by

$$\left|\psi\right\rangle = \cos\tfrac{\theta}{2}\left|+\right\rangle + \sin\tfrac{\theta}{2}\,e^{i\phi}\left|-\right\rangle.$$

For a statistical ensemble of spins where all directions $\vec{u}(\theta,\phi)$ are equally likely show:

$$\overline{\left\langle \hat{M}_x \right\rangle} = \overline{\left\langle \hat{M}_y \right\rangle} = \overline{\left\langle \hat{M}_z \right\rangle} = 0.$$

(which represents an unpolarized spin system).

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3. **The Bloch Equation Revisited**

Consider a spin-1/2 particle with associated magnetic moment operators $\hat{\mu}_x$, $\hat{\mu}_y$, and $\hat{\mu}_z$ (written more compactly as $\hat{\vec{\mu}} = \gamma\hbar\hat{\vec{I}}$).

a) While the operator $\hat{\vec{\mu}}$ is independent of time, the expected value $\left\langle \hat{\vec{\mu}} \right\rangle$ is, in general, time varying. Show:

$$i\frac{d}{dt}\left\langle \hat{\vec{\mu}} \right\rangle(t) = \left\langle \left[\hat{\vec{\mu}}, \hat{H} \right] \right\rangle$$

where $\hat{H}$ is the Hamiltonial operator of the system.

b) When placed in a magnetic field $\vec{B}(t)$, $\hbar\hat{H}(t) = -\hat{\vec{\mu}} \cdot \vec{B}(t)$.

Show:

$$\frac{d}{dt}\left\langle \hat{\vec{\mu}} \right\rangle(t) = \gamma \left\langle \hat{\vec{\mu}} \right\rangle(t) \times \vec{B}(t)$$

(hint: use the commulator relationships for $\hat{I}_x$, $\hat{I}_y$, and $\hat{I}_z$).

Hence, the expected value of $\hat{\vec{\mu}}$ obeys the classical Bloch equation (ignoring relaxation)!

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4. **Basis sets in Liouville space.**

Prove that if the kets $|f_i\rangle$ form an orthonormal basis for an n-dimensional Hilbert space, then the transition operators $\hat{T}_{ij} = |f_i\rangle\langle f_j|$ constitute an orthonormal basis in the corresponding Liouville space.