1. The Pauli matrices

Let $|\varphi_{x_i}\rangle$, $|\varphi_{y_i}\rangle$, and $|\varphi_{z_i}\rangle$ for $i = 1, 2$ be the eigenkets for $\hat{I}_x$, $\hat{I}_y$, and $\hat{I}_z$.

The Pauli matrices, expressed in the $|\varphi_{z_i}\rangle$ basis, are given by:

$$
\hat{I}_x = \frac{1}{2} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \quad \hat{I}_y = \frac{1}{2} \begin{bmatrix}
0 & -i \\
i & 0
\end{bmatrix} \quad \hat{I}_z = \frac{1}{2} \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
$$

a) Find the corresponding eigenvectors and eigenvalues for $\hat{I}_x$, $\hat{I}_y$, and $\hat{I}_z$.

b) Show by direct matrix calculation that:

$$
\sum_{i=1}^{2} |\varphi_{x_i}\rangle \langle \varphi_{x_i}| = \sum_{i=1}^{2} |\varphi_{y_i}\rangle \langle \varphi_{y_i}| = \sum_{i=1}^{2} |\varphi_{z_i}\rangle \langle \varphi_{z_i}| = \hat{E} \quad \text{(identity operator)}
$$

c) Find $[\hat{I}_z, \hat{I}_x]$, $[\hat{I}_z, \hat{I}_y]$, and $[\hat{I}_x, \hat{I}_y]$.

d) Express $\hat{I}_x$, $\hat{I}_y$, and $\hat{I}_z$ in the $|\varphi_{y_i}\rangle$ basis.
2. **Stern-Gerlach Experiment**

The $x$, $y$, and $z$ components of the intrinsic angular momentum (spin) of a spin-1/2 particle are given by the cyclically commuting operators:

\[ \hat{I}_x, \hat{I}_y, \text{ and } \hat{I}_z; \]

each with eigenvalues $\pm 1/2$

Consider an arbitrary unit vector $\vec{u}$.

a) What are the eigenvalues of $\hat{I}_u$, the operator corresponding to the spin in the $\vec{u}$ direction?

b) What is the matrix form of $\hat{I}_u$ expressed in the $\{+\},\{-\}$ basis, where $|+\rangle$ and $|-\rangle$ are the eigenkets of $\hat{I}_z$ corresponding to eigenvalues $+1/2$ and $-1/2$ respectively?
c) The general form of a ket for single spin-1/2 particle is

\[ |\psi\rangle = \alpha|+\rangle + \beta|-\rangle \text{ where } |\alpha|^2 + |\beta|^2 = 1. \]

Can you find \( \alpha \) and \( \beta \) such that \( |\psi\rangle \) represents an unpolarized spin, i.e.

\[ \langle \hat{I}_x \rangle = \langle \hat{I}_y \rangle = \langle \hat{I}_z \rangle = 0? \]

d) Consider a collection of spin-1/2 nuclei each described by the same ket

\[ |s_p\rangle = \frac{1}{\sqrt{3}} (|s_x\rangle + |s_y\rangle) \]

where \( |s_x\rangle \) and \( |s_y\rangle \) are spin states oriented along the +x and +y directions respectively. This system is said to be in a “pure” state.

Show \( |s_p\rangle \) is normalized.

Find \( \langle \hat{M}_x \rangle, \langle \hat{M}_y \rangle, \) and \( \langle \hat{M}_z \rangle. \)
e) Silver atoms leaving the furnace in the Stern-Gerlach experiment can be polarized in any direction. It can be shown that the state of a spin polarized in an arbitrary direction $\tilde{u}$ is given by

$$|\psi\rangle = \cos\frac{\theta}{2} |+\rangle + \sin\frac{\theta}{2} e^{i\phi} |-\rangle.$$ 

For a statistical ensemble of spins where all directions $\tilde{u}(\theta,\phi)$ are equally likely show:

$$\langle \hat{M}_x \rangle = \langle \hat{M}_y \rangle = \langle \hat{M}_z \rangle = 0.$$ 

(which represents an unpolarized spin system).
3. The Bloch Equation Revisited
Consider a spin-1/2 particle with associated magnetic moment operators $\hat{\mu}_x$, $\hat{\mu}_y$, and $\hat{\mu}_z$ (written more compactly as $\hat{\mu} = \gamma \hbar \hat{I}$).

a) While the operator $\hat{\mu}$ is independent of time, the expected value $\langle \hat{\mu} \rangle$ is, in general, time varying. Show:

$$i \frac{d}{dt} \langle \hat{\mu} \rangle(t) = \left\langle \left[ \hat{\mu}, \hat{H} \right] \right\rangle$$

where $\hat{H}$ is the Hamiltonian operator of the system.

b) When placed in a magnetic field $\vec{B}(t)$, $\hbar \hat{H}(t) = -\hat{\mu} \cdot \vec{B}(t)$.

Show:

$$\frac{d}{dt} \langle \hat{\mu} \rangle(t) = \gamma \langle \hat{\mu} \rangle(t) \times \vec{B}(t)$$

(hint: use the commutator relationships for $\hat{I}_x$, $\hat{I}_y$, and $\hat{I}_z$).

Hence, the expected value of $\hat{\mu}$ obeys the classical Bloch equation (ignoring relaxation)!
Problem Set #3  
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4. **Basis sets in Liouville space.**

Prove that if the kets $|f_i\rangle$ form an orthonormal basis for an $n$-dimensional Hilbert space, then the transition operators $\hat{T}_{ij} = |f_i\rangle \langle f_j|$ constitute an orthonormal basis in the corresponding Liouville space.