# Signal Calculations

- Bloch Equations and Matrix Calculations
- Extended Phase Graphs
- Examples (both)

# Bloch Equation Matrix Simulations

- Basic Bloch Equation
- Bloch Equation with $B_1$ / rotating frame
- Basic matrix simulations / Hard Pulse Approx.
- Many-spin simulations:  Brute force
- Bloch-McConnell Equation with Exchange
- Bloch-Torrey Equations (McNab?)

# Bloch Equation

- Basic Bloch Equation:

$$\frac{dM}{dt} = M \times \gamma B - \frac{M_{xy}}{T_2} + \frac{M_0 - M_z}{T_1}$$

- In Matrix form with:

$$M = \left[ \begin{array}{c} M_x \\ M_y \\ M_z \end{array} \right]$$

- Becomes:

$$\frac{dM}{dt} = \left[ \begin{array}{ccc} -1/T_2 & \gamma B_z & -\gamma B_y \\ -\gamma B_z & -1/T_2 & \gamma B_x \\ \gamma B_y & -\gamma B_x & -1/T_1 \end{array} \right] M + \left[ \begin{array}{c} 0 \\ 0 \\ M_0/T_1 \end{array} \right]$$

*Standard right-handed cross product*

# Relaxation

- Over time period $\tau$

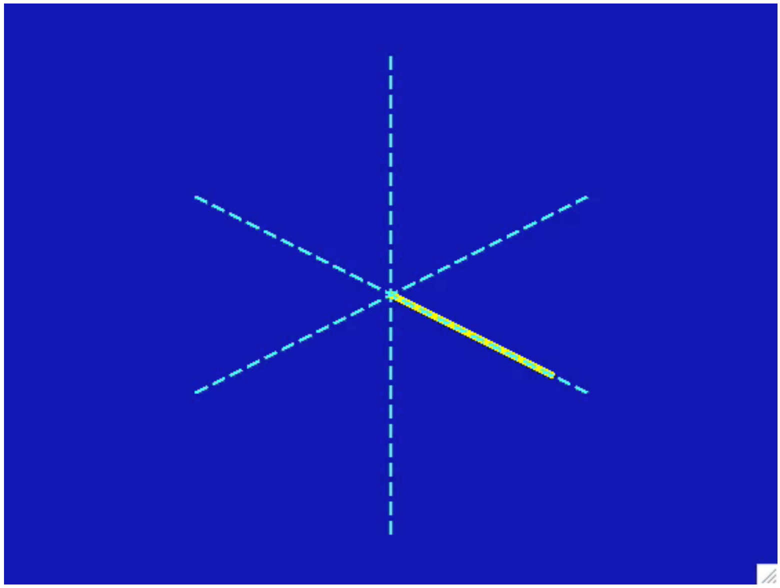
$$E_1 = e^{-\tau/T_1}$$

$$E_2 = e^{-\tau/T_2}$$

$$M' = \underbrace{\begin{bmatrix} E_2 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_1 \end{bmatrix}}_{A} M + \underbrace{\begin{bmatrix} 0 \\ 0 \\ M_0(1-E_1) \end{bmatrix}}_{B}$$

*(See relax.m)*

```
>> [AB] = relax(0.5,0.5,0.1,1)

  AB =
     0.0067         0         0         0
          0    0.0067         0         0
          0         0    0.3679    0.6321
```

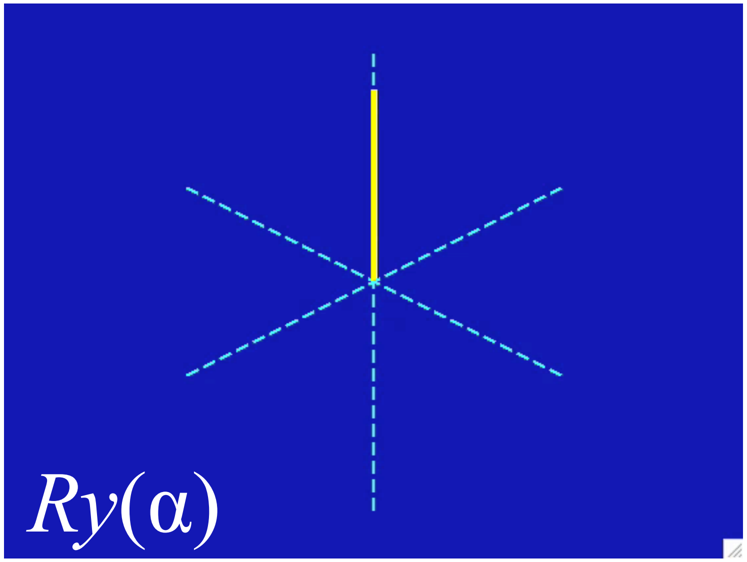


# RF Rotations

- For a flip angle α

$$M' = R_x M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} M$$

$$M' = R_y M = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} M$$




$$\alpha = \gamma B_1 \tau$$

- *Rotations are left-handed*
- *(also achieved with negative γ)*
- *See xrot.m and yrot.m*

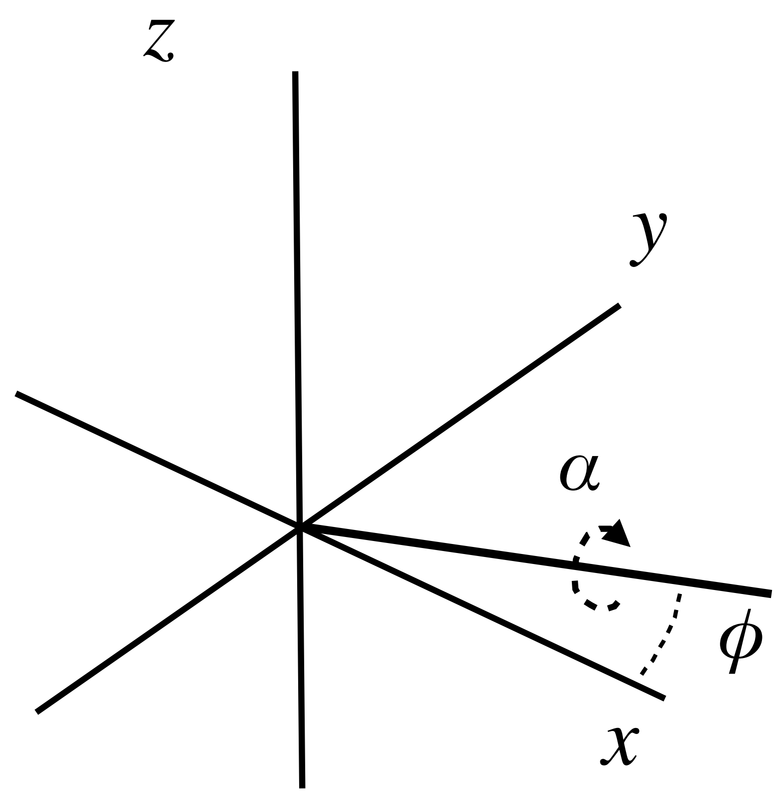
```
>> A = xrot(90)

A =

     1.00          0          0
        0          0       1.00
        0      -1.00          0
```

# RF Rotations (Arbitrary $B_1$ phase)

$$R_\phi = \begin{bmatrix} \cos^2\phi + \sin^2\phi\cos\alpha & \cos\phi\sin\phi(1-\cos\alpha) & -\sin\phi\sin\alpha \\ \cos\phi\sin\phi(1-\cos\alpha) & \sin^2\phi + \cos^2\phi\cos\alpha & \cos\phi\sin\alpha \\ \sin\phi\sin\alpha & -\sin\alpha\cos\phi & \cos\alpha \end{bmatrix}$$

$$\phi = \tan^{-1}(B_y/B_x)$$

- Rotation axis in x-y plane
- Note $R_\phi$ is just $R_z(-\phi)R_x(\alpha)\,R_z(\phi)$

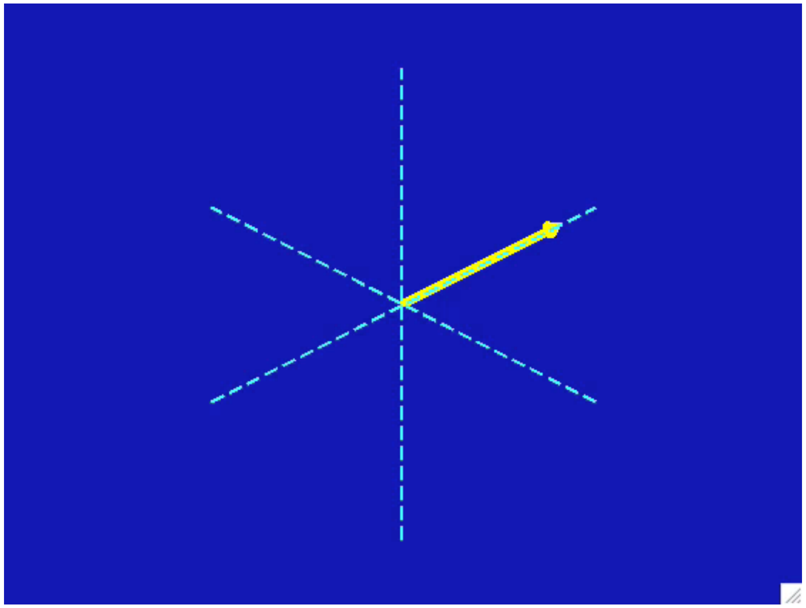



*(See throt.m)*

# Gradient / $\Delta B_0$ Rotations

$$M' = R_z M = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} M$$

$$\theta = \gamma(G \cdot \overrightarrow{r} + \Delta B_0)\tau$$

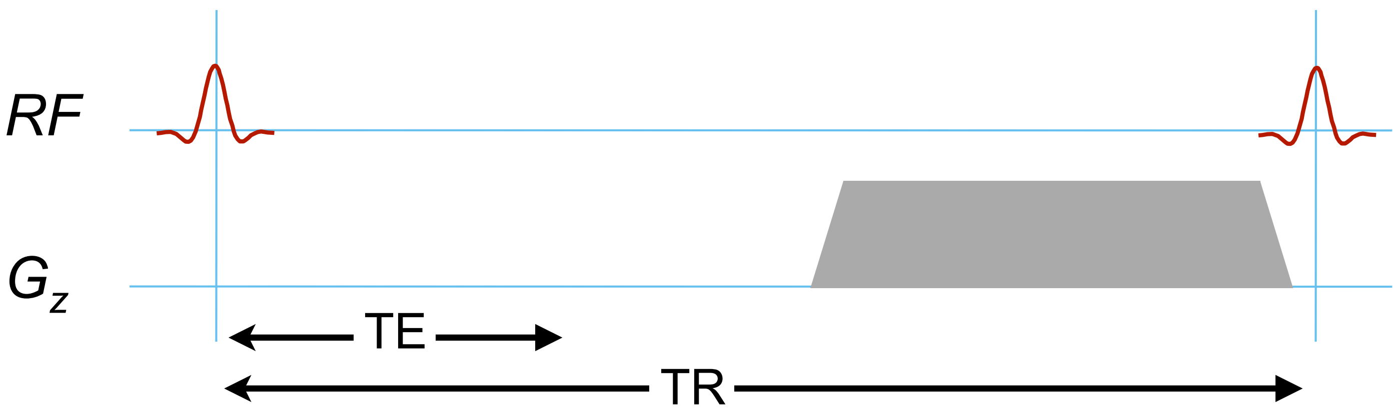


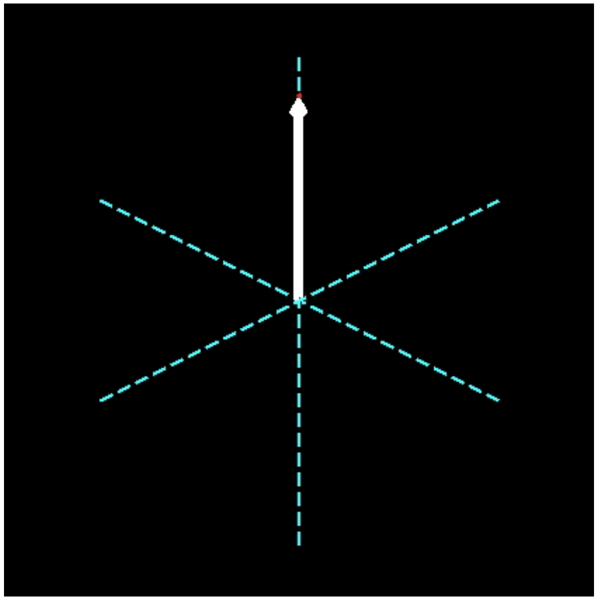
- *Rotations are left-handed*
- *(also achieved with negative $\gamma$)*
- *See zrot.m*

# Perfect Spoiling

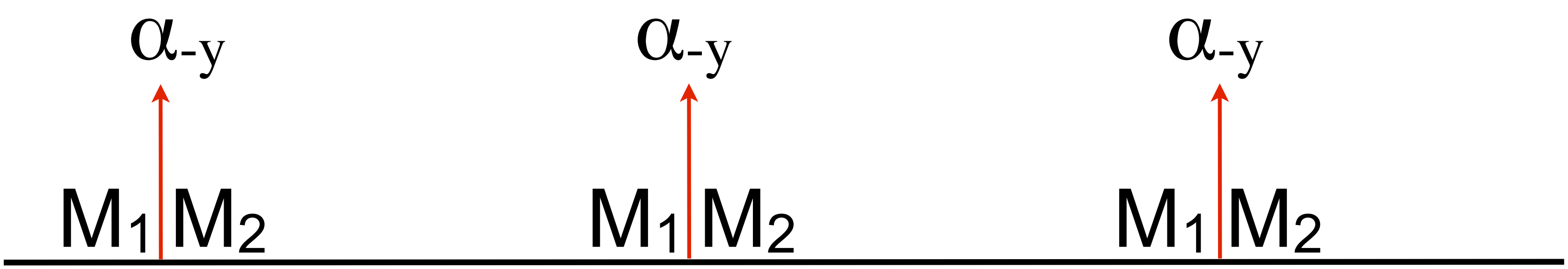
- Explicitly set transverse magnetization to zero

$$M' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} M$$



# Example: Excitation/Recovery




$$M_2 = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} M_1$$

*Neglect residual transverse magnetization*

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_1 \end{bmatrix} M_2 + \begin{bmatrix} 0 \\ 0 \\ m_0(1-E_1) \end{bmatrix}$$

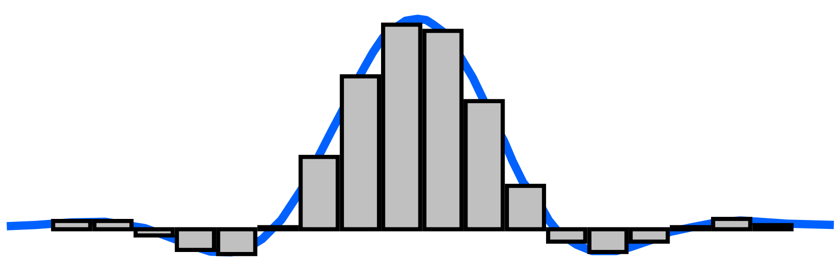
$$M_1 = E_1 \cos\alpha \, M_1 + m_0(1-E_1)$$

$$\boxed{M_1 = \frac{m_0(1-E_1)}{1-E_1\cos\alpha}}$$

*Recall Prior Example!*

# Overlapping RF/Gradients?

- z rotations and relaxation commute
- RF rotations <u>do not</u> commute with others
- **Hard-Pulse Approximation:**
  - Small rotations/relaxation can be applied sequentially
  - Break pulses into very short segments
  - Aside:  Basis for Variable Rate Selective Excitation
- **Alternatively, calculate arbitrary rotations**



# Matrix Propagation

- Rotations / Relaxation: $M' = AM+B$
  - $M_1 = A_1 M_0 + B_1$ *($M_0$ is starting M, not equilibrium M!)*
  - $M_2 = A_2 M_1 + B_2$ ..... $M_n$ *(n "operations")*
  - *Propagation: multiply A's, sum B's after multiplying by all successive A's*
  - $M_2 = (A_2 A_1) M_0 + A_2 B_1 + B_2$

$$A = \prod_{i=n}^{1} A_i$$

$$B = \sum_{i=1}^{n} \left( \prod_{j=n}^{i+1} Aj \right) B_i$$

*Example (n=3)*

$$A = A_3 A_2 A_1$$
$$B = A_3 A_2 B_1 + A_3 B_2 + B_3$$

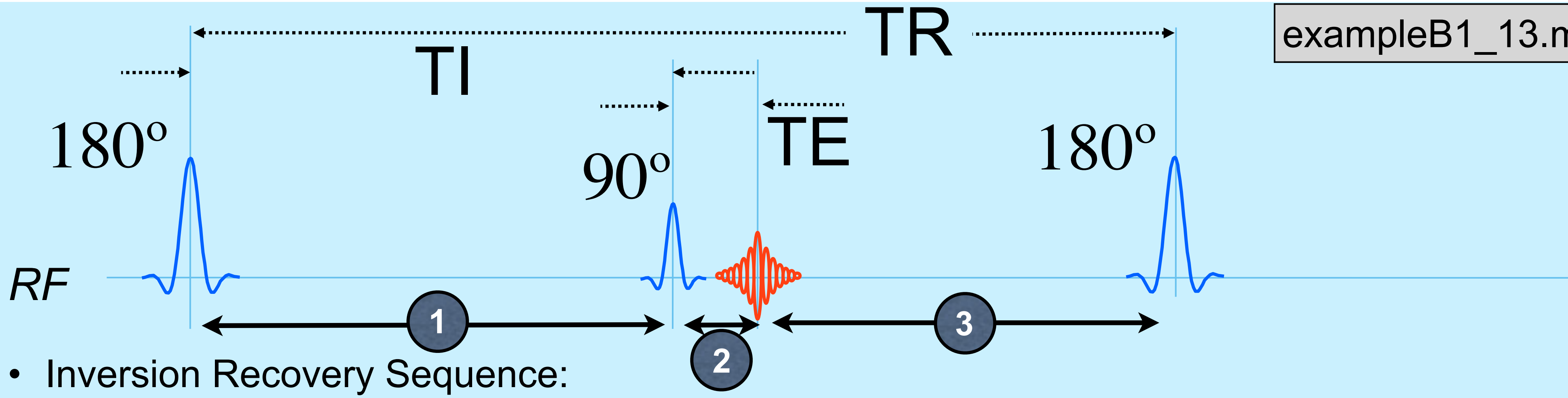
*(See abprop.m)*

# Steady States

- Propagation over 1 TR: $M_{n+1} = AM_n + B$
- Steady State: $M_{n+1} = M_n$
- Combine: $M_{ss} = AM_{ss} + B \quad = (I-A)^{-1}B$
  - As long as there is relaxation, there is a steady state, since eigenvalues of A are less than one in magnitude

# Short-TR IR Signal (Again!)

exampleB1_13.m




- Inversion Recovery Sequence:
  - TR = 1s, TI = 0.5s, TE=50ms
- What is the signal for $T_1$=0.5s, $T_2$=100ms?
  - "Operations"
  - $M_{180} = R_x(180)M_{TR}$
  - $M_{90} = R_x(90)E(0.5s)M_{180}$
  - $M_{TE} = E(0.05s)M_{90}$
  - $M_{TR} = E(0.45s = 1\text{-}0.5\text{-}0.05s)M_{TE}$

```
>> A1 = diag([E2a E2a E1a]) * Rx(180);
>> B1 = [0;0;1-E1a];
>> A2 = diag([E2b E2b E1b]) * Rx(90);
>> B2 = [0;0;1-E1b];
>> A3 = diag([E2c E2c E1c]);
>> B3 = [0;0;1-E1c];

>> A = A2*A1*A3;
>> B = B2+A2*(B1+A1*B3);

>> M = inv(eye(3)-A)*B
     [ 0;  0.2424;  0.0952 ]
```

# Compact Simulation: abprop.m

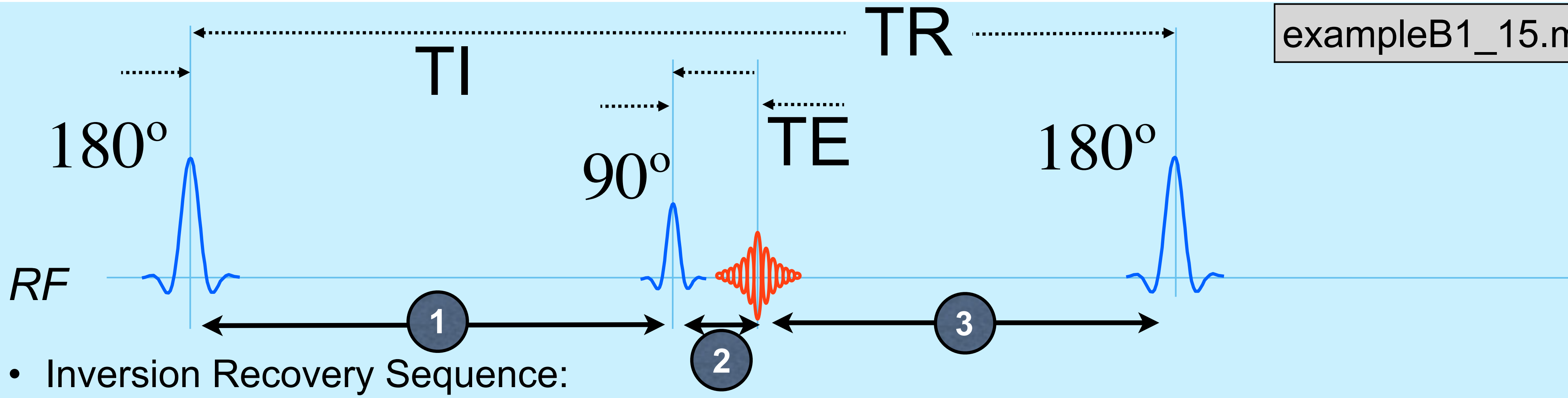
- Propagate spins through series of A,B matrices
- Compact way to simulate sequences

```
function [A,B,mss] = abprop(A1,B1,A2,B2,A3,B3,...)

  If mss is provided, the steady-state is calculated.

  If an Ai is 3x4, then it is assumed to be [Ai Bi]

  If a Bi vector is omitted (the next argument is 3x3
  or 3x4, it is assumed to be zero.
```

- abprop($A_1$,$B_1$, [$A_2$ $B_2$], $A_3$, $A_4$) *(Here $B_3 = B_4 = 0$)*

# Short-TR IR Signal (Compact)

exampleB1_15.m

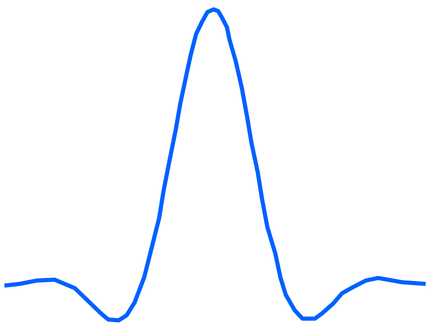



- Inversion Recovery Sequence:
  - TR = 1s, TI = 0.5s, TE=50ms
- What is the signal for $T_1$=0.5s, $T_2$=100ms?

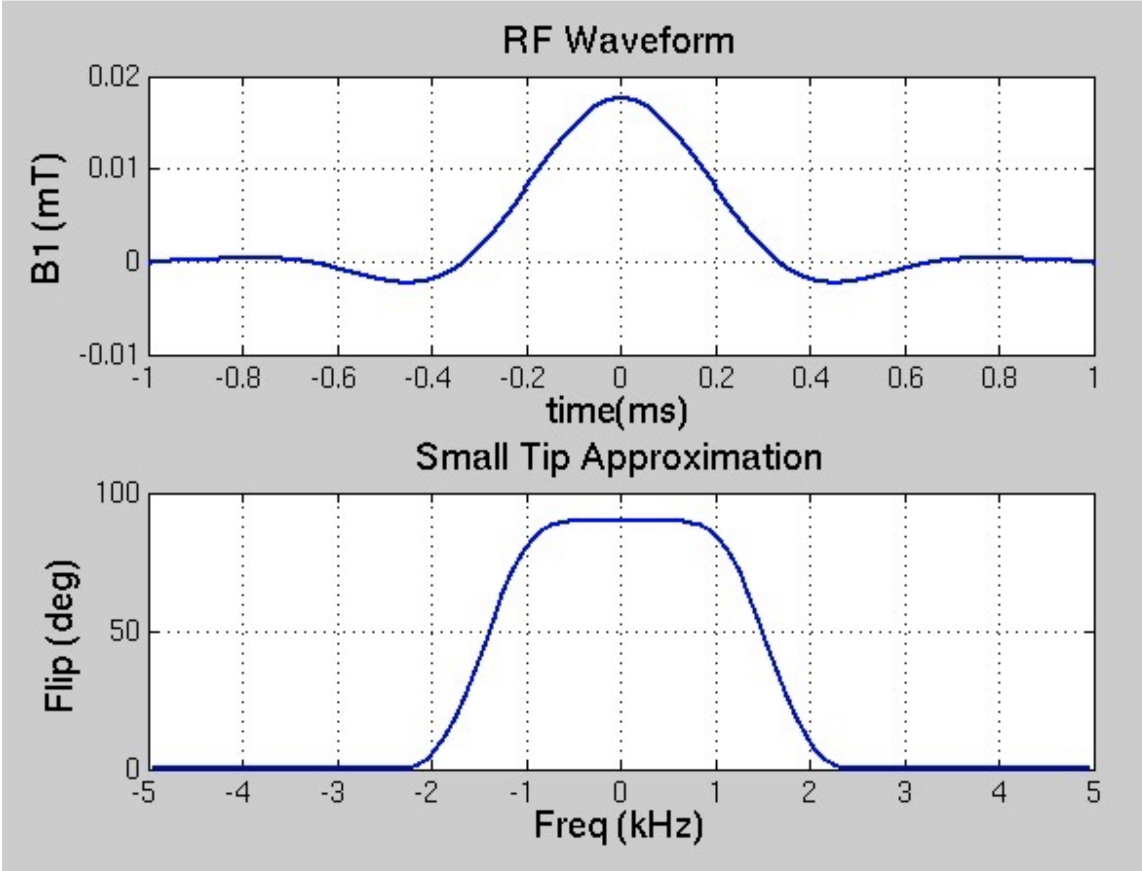
  - "Operations"
  - $M_{180} = R_x(180)M_{TR}$
  - $M_{90} = R_x(90)E(0.5s)M_{180}$
  - $M_{TE} = E(0.05s)M_{90}$
  - $M_{TR} = E(0.45s = 1\text{-}0.5\text{-}0.05s)M_{TE}$

```
>> [A,B,Mss] = abprop(
       relax(TR-TE-TI,T1,T2,1),  …
       xrot(180), ...
       relax(TI,T1,T2,1),  …
       xrot(90), …
       relax(TE,T1,T2,1));

>> Mss
       [ 0;  0.2424;  0.0952 ]
```

# Example



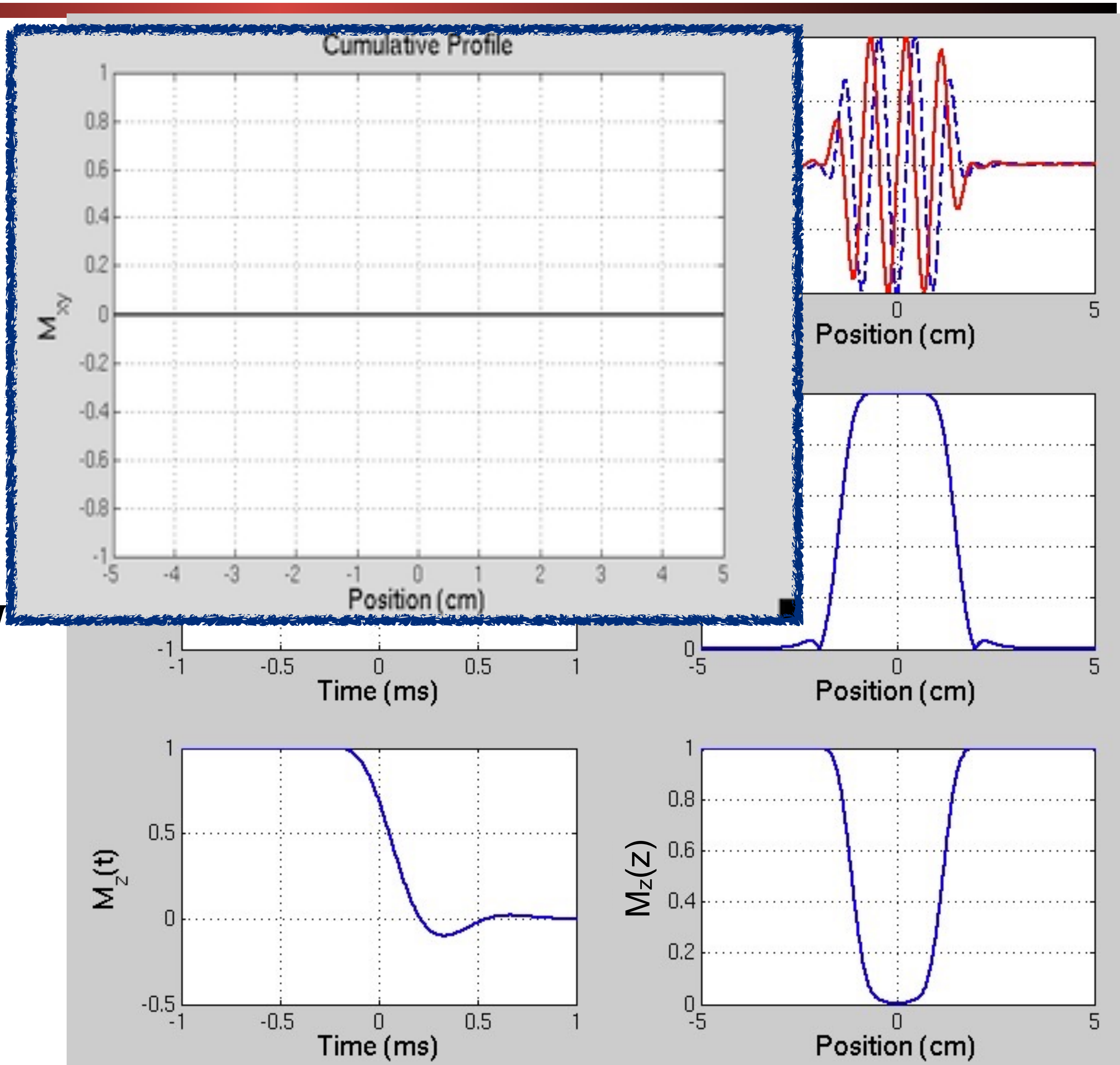
- $90_{-y}$ Excitation pulse
  - Time samples of 4μs
  - 3 sinc cycles
  - 2ms duration
  - Area of 5.9 μT*ms
  - BW ~ 3 kHz
  - 2.3 mT/m gradient (1kHz/cm)



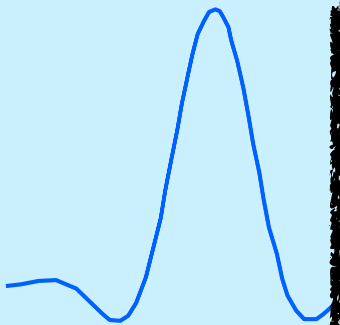
# Simulation

- Loop over z
  - Define $R_z$
  - Loop over t
    - $M' = R_z R_{-y}(t) M$
- Plot M over time and space

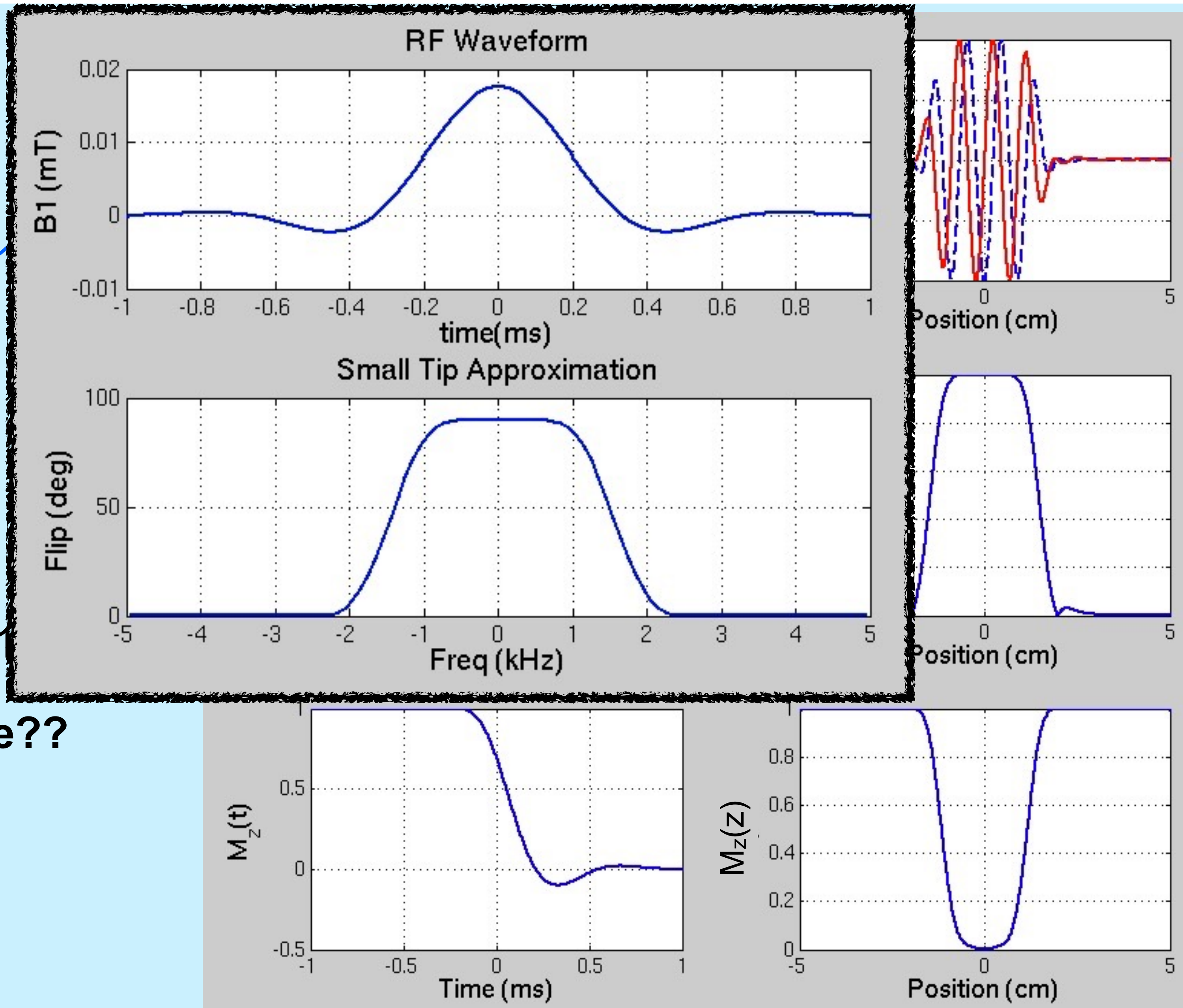
exampleB1_17.m



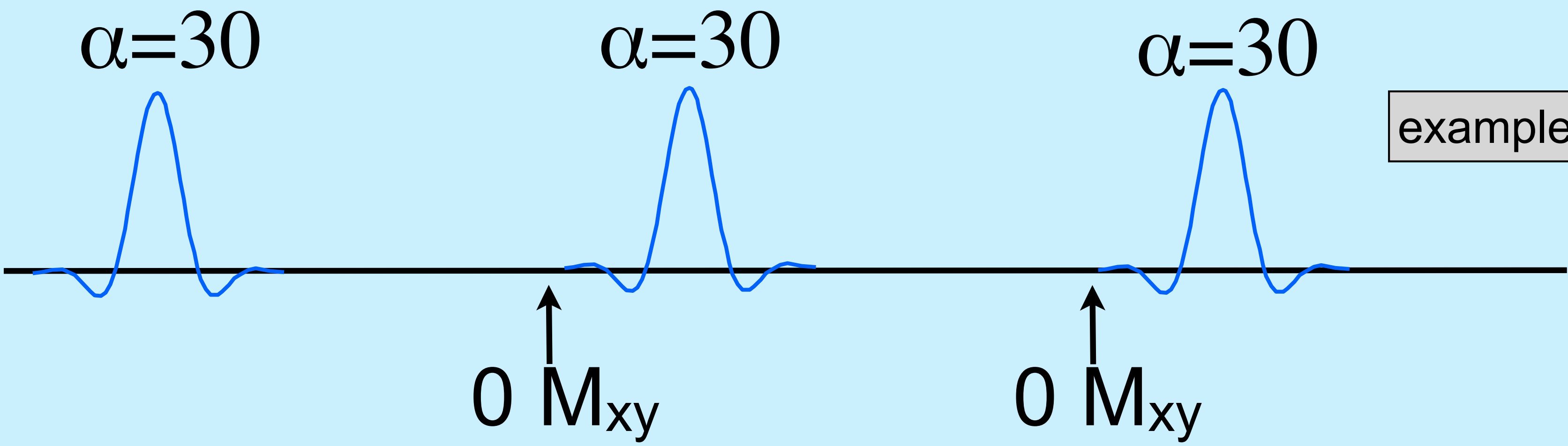
# Example (With Off-Resonance)



- 90 Excitation pulse
- BW ~ 3 kHz
- 2.3 mT/m gradient (1
- **2kHz off-resonance??**



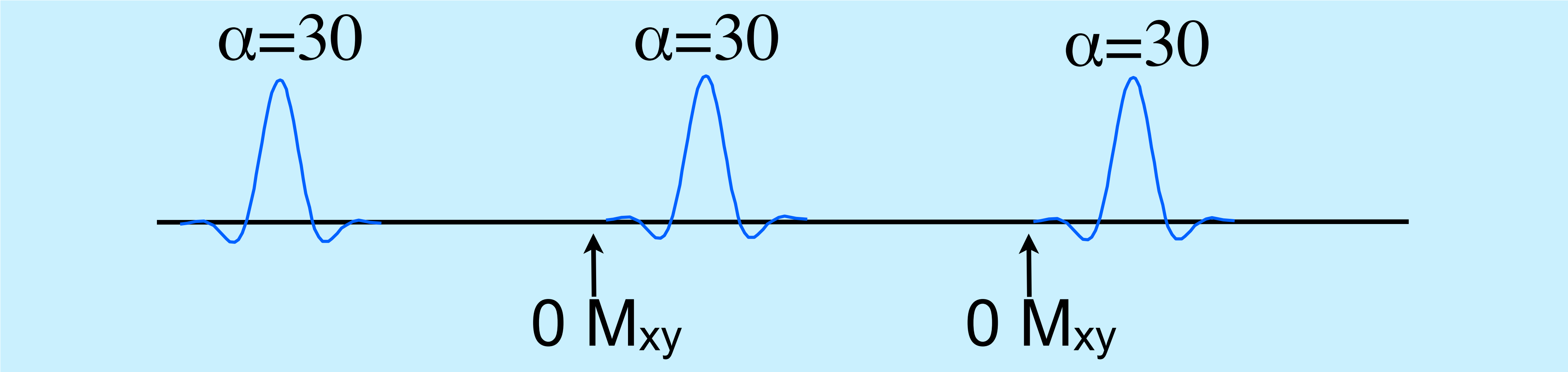
# Excitation Recovery (Real Pulse)

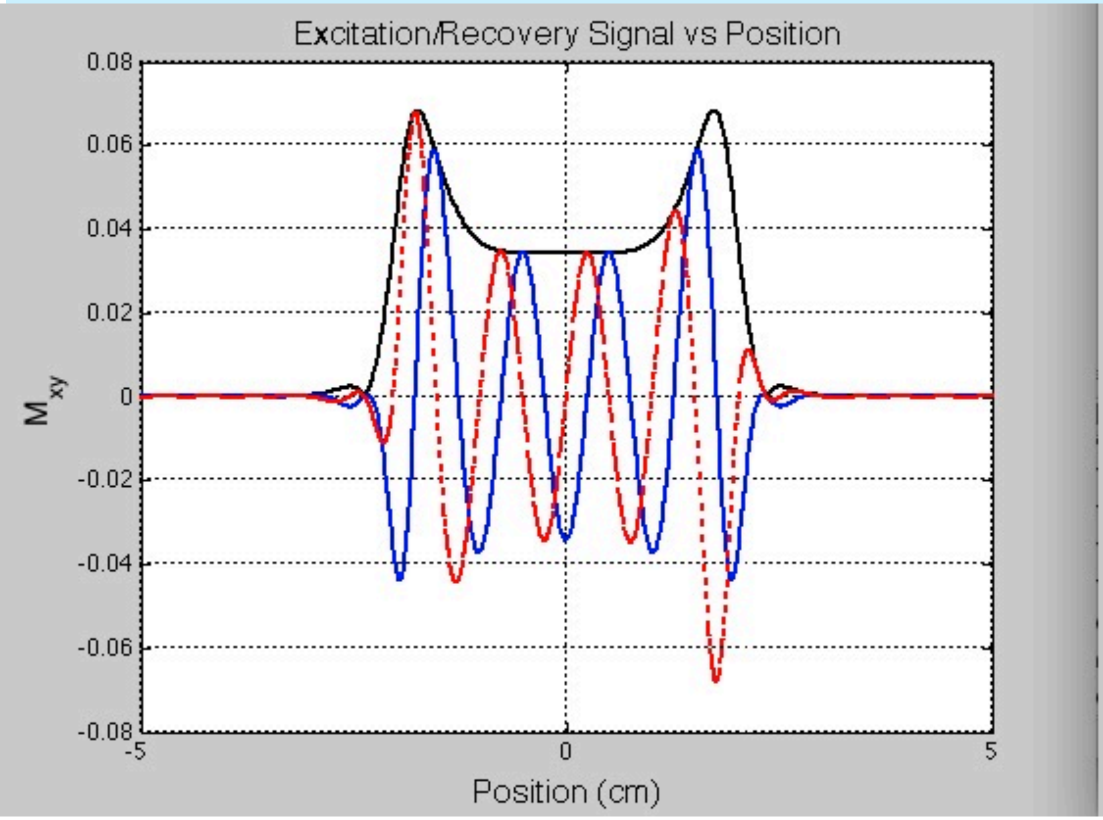


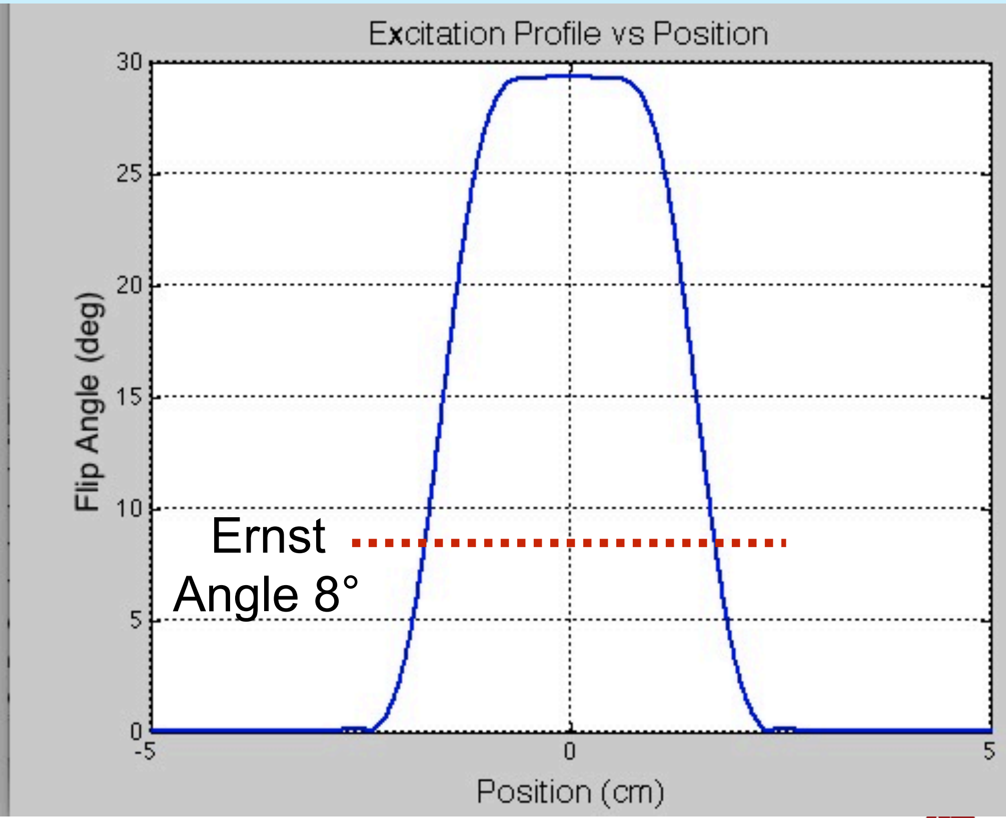

exampleB1_19.m

- Simulate full pulse and position
- Perfect spoiling ("keep only $M_z$" matrix)
- Matrix propagation to calculate steady-state at each position: E() = relaxation
  - End of RF to TR: Spoil, $E(TR-T_{RF})$
  - Over RF: [ $E(\tau)R_z(\gamma Gz\tau)\ R_{-y}(t,\tau)$ ] at each interval

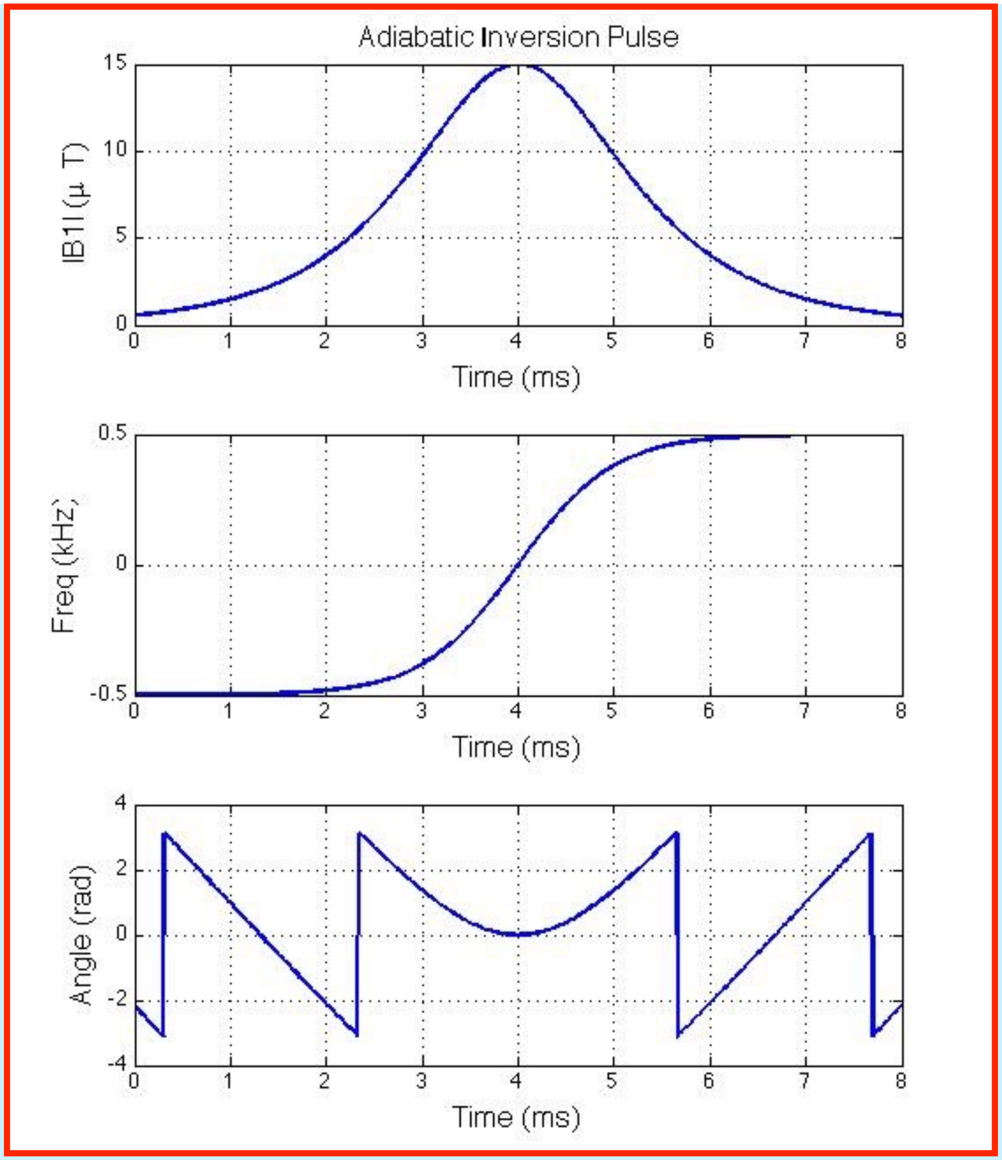
# Excitation Recovery (Real Pulse)

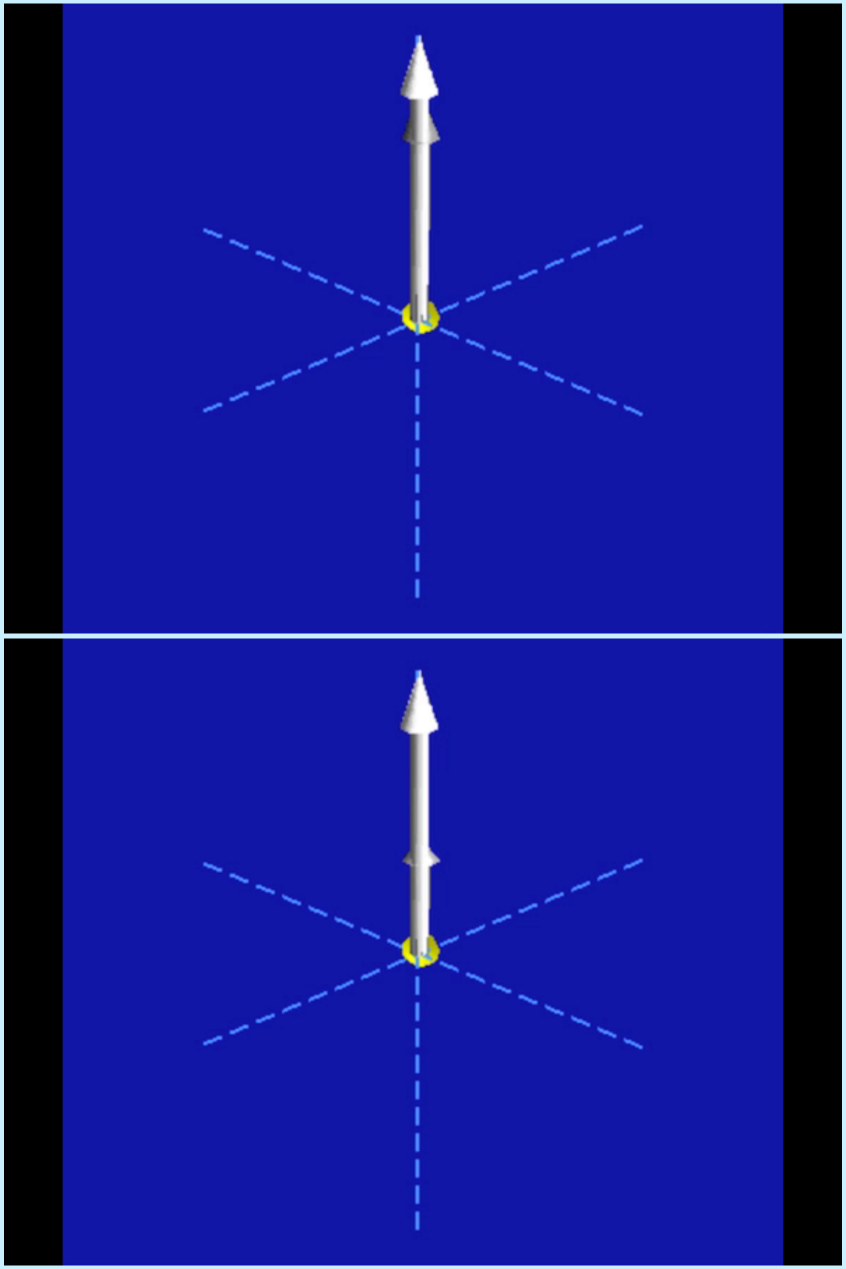


# Example: Adiabatic Pulse



*(See adiabatic.m,*
*[b1,freq,phase] = adiabatic(.15,1000,1000,0.008,.000004) )*

# Exchange: Bloch-McConnell Equation

- $M^a$ and $M^b$ are magnetizations in "exchanging" pools:
  - $\tau_a$ and $\tau_b$ are resident times

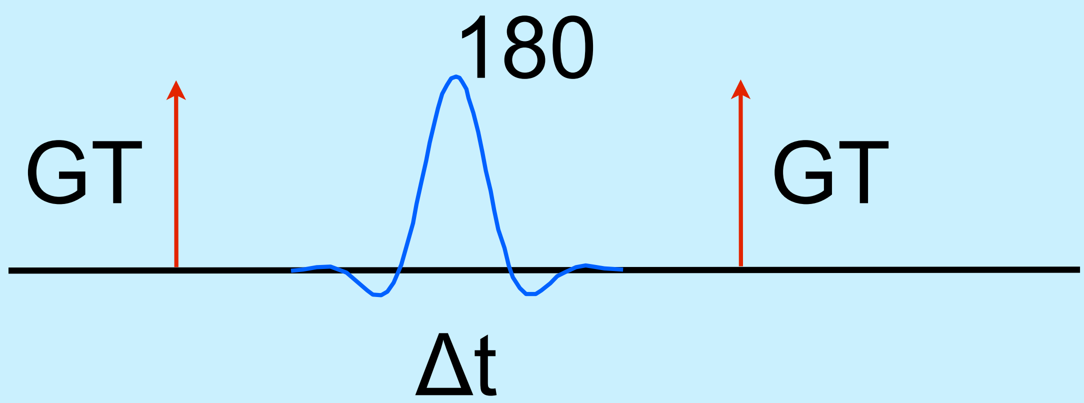
$$\frac{dM}{dt} = \begin{bmatrix} -1/T_2^a - 1/\tau_a & \gamma_a B_z & -\gamma_a B_y & 1/\tau_b & 0 & 0 \\ -\gamma_a B_z & -1/T_2^a - 1/\tau_a & \gamma_a B_x & 0 & 1/\tau_b & 0 \\ \gamma_a B_y & -\gamma_a B_x & -1/T_1^a - 1/\tau_a & 0 & 0 & 1/\tau_b \\ 1/\tau_a & 0 & 0 & -1/T_2^b - 1/\tau_b & \gamma_b B_z & -\gamma_b B_y \\ 0 & 1/\tau_a & 0 & -\gamma_b B_z & -1/T_2^b - 1/\tau_b & \gamma_b B_x \\ 0 & 0 & 1/\tau_a & \gamma_b B_y & -\gamma_b B_x & -1/T_1^b - 1/\tau_b \end{bmatrix} M + \begin{bmatrix} 0 \\ 0 \\ M_0^a/T_1^a \\ 0 \\ 0 \\ M_0^b/T_1^b \end{bmatrix}$$

$$M = \begin{bmatrix} M_x^a \\ M_y^a \\ M_z^a \\ M_x^b \\ M_y^b \\ M_z^b \end{bmatrix}$$

**Single-Pool Bloch Equation:**

$$\frac{dM}{dt} = \begin{bmatrix} -1/T_2 & \gamma B_z & -\gamma B_y \\ -\gamma B_z & -1/T_2 & \gamma B_x \\ \gamma B_y & -\gamma B_x & -1/T_1 \end{bmatrix} M + \begin{bmatrix} 0 \\ 0 \\ M_0/T_1 \end{bmatrix}$$

# Challenge: Diffusion




- 1D Gaussian Diffusion: $\Delta l = \sqrt{2D\Delta t}$
- Imagine a sequence with 2 gradients of area GT, with a 180 refocusing pulse between.
- What is the expected value of the spin echo signal as a function of D, Δt, GT, ignoring $T_2$?

# Summary

- Bloch equation:  Rotations and Relaxation
- Consider M as a 3x1 vector
  - Rotations ~ Simple multiplier
  - Relaxation ~ $M' = AM+B$
  - Propagate Effects like "Operators"
- Brute force simulations by looping:
  - Time, Position, Frequency, etc