Rad229 – MRI Signals and Sequences

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Lecture-9A — Gradient-Echo Sequences

Mathematics of Balanced SSFP (bSSFP)

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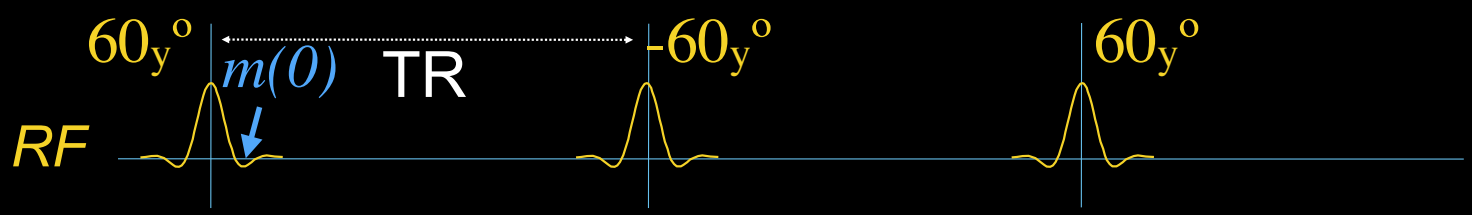
Learning Objectives

- Describe a matrix formulation of the bSSFP signal
- Describe a geometrical forumulation of the signal
- Explain transient signals mathematically

Matrix Derivation of bSSFP Signal



- Sequence: $E_{Relaxation} - R_z(180^o) - R_{-y}(\alpha)$
  - Note $R_z(180^o)$ lets us do this over only 1TR
- $m(0) = R_{-y}(\alpha)\, R_z(180^o)\, E\, m(0) + R_{-y}(\alpha)\, [0;0;1-E_1]$
  - Note: no M_y

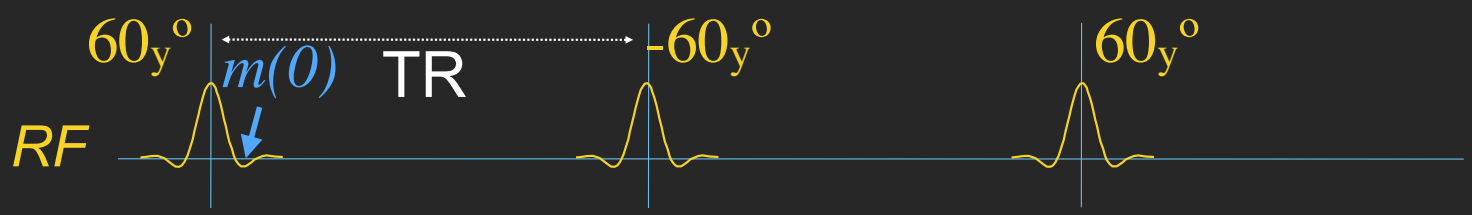
$$\begin{bmatrix} 1+E_2\cos\alpha & 0 & -E_1\sin\alpha \\ 0 & 1+E_2 & 0 \\ -E_2\sin\alpha & 0 & 1-E_1\cos\alpha \end{bmatrix} m(0) = \begin{bmatrix} \sin\alpha(1-E_1) \\ 0 \\ \cos\alpha(1-E_1) \end{bmatrix}$$

Some algebra…

$$m(0) = \frac{1-E_1}{1+\cos\alpha(E_2-E_1)-E_1E_2} \begin{bmatrix} \sin\alpha \\ 0 \\ E_2+\cos\alpha \end{bmatrix}$$

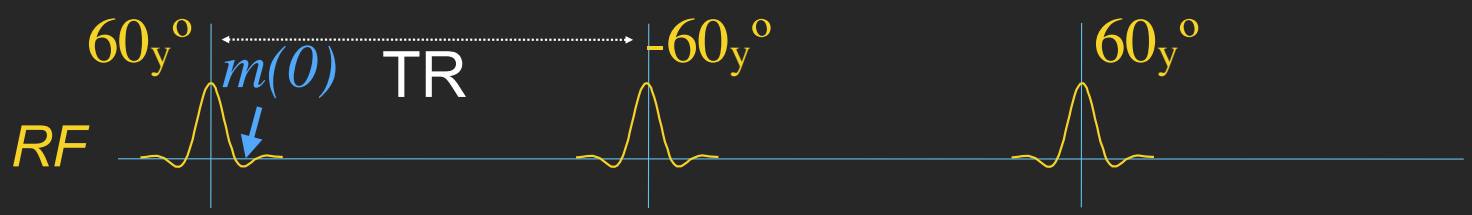
The matrix derivation gives the exact steady-state vs flip angle for T_1 and T_2

Question 1: Steady-state Elevation Angle

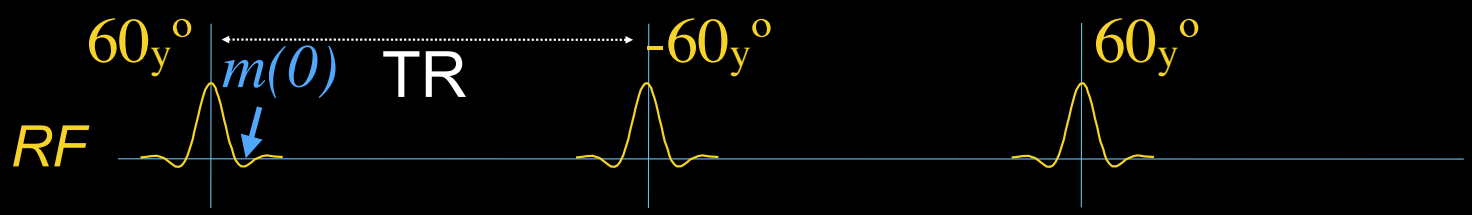


This matrix deriviation passes some basic checks.

Question 2: Short-T_2?



Matrix Derivation of bSSFP Signal with off-resonance

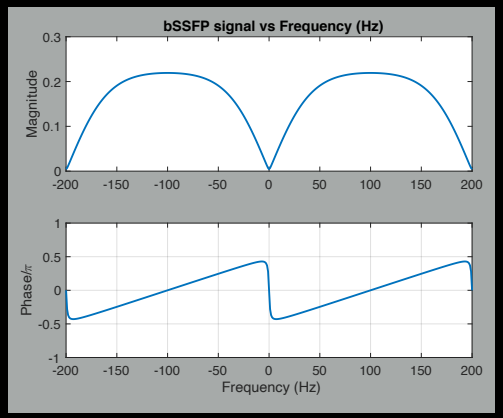


- Sequence: $E_{Relaxation} - R_z(180^o + \phi) - R_{-y}(\alpha)$
  - Note $R_z(180^o)$ still lets us do this over only 1TR

$$m(0) = \frac{ae^{i\phi} + b}{c\cos\phi + d}$$

$$a = -(1 - E_1)E_2 \sin\alpha$$
$$b = (1 - E_1)\sin\alpha$$
$$c = E_2(E_1 - 1)(1 + \cos\alpha)$$
$$d = 1 - E_1\cos\alpha - (E_1 - cos\alpha)E_2^2$$

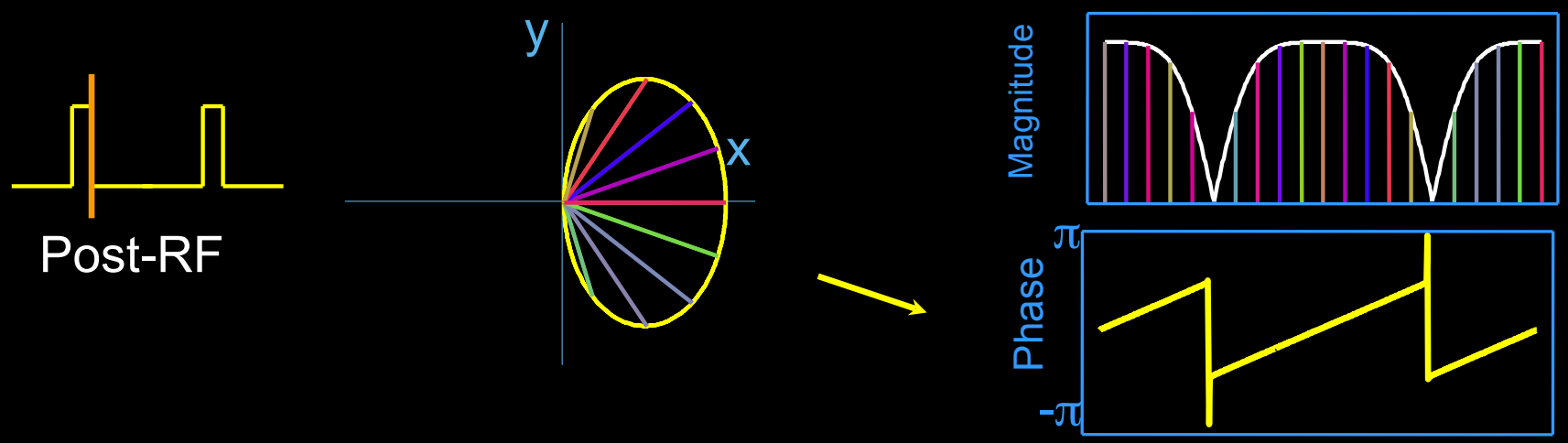
$$E_{1,2} = e^{-T_{1,2}/TR}$$



The equation is for signal after the RF pulse, with off-resonance precession. lec9_01.m

Matrix Signal: Consideration of Precession and T_2-decay

- Derived on-resonance, but immedately after RF
  - Magnetization is refocused at $TE = TR/2$
  - Magnetization has linear phase at $TE = 0$
- (Small) T_2 decay to TE of e^{-TE/T_2}



The signal immediately after the RF has linear phase, and decays slightly over the TR

"Ellipsoid" Signal Derivation

- Start with ellipsoid:

$$\left(M_z - \frac{M_0}{2}\right)^2 + \frac{M_x^2 + M_y^2}{T_2/T_1} = \left(\frac{M_0}{2}\right)^2$$

- Substitute $Mz = M\cos(\beta/2)$ and $M_{xy} = M\sin(\beta/2)$:

$$\left[M\cos(\beta/2) - \frac{M_0}{2}\right]^2 + \frac{M^2\sin^2(\beta/2)}{T_2/T_1} = \left[\frac{M_0}{2}\right]^2$$

- Multiply out, bring up T_2/T_1:

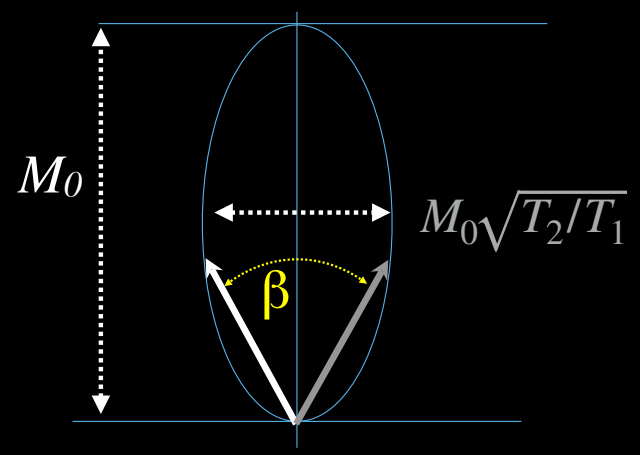
$$M^2\cos^2(\beta/2) + (T_1/T_2)M^2\sin^2(\beta/2) = MM_0\cos(\beta/2)$$

- Divide out M, rearrange and multiply by $\sin(\beta/2)$:

$$M\sin(\beta/2) = \frac{M_0\cos(\beta/2)\sin(\beta/2)}{\cos^2(\beta/2) + (T_1/T_2)M^2\sin^2(\beta/2)}$$

- Divide terms along denominator by numerator:

$$M_{xy} = \frac{M_0}{\cot(\beta/2) + (T_1/T_2)\tan(\beta/2)}$$



- Signal drops with increasing T_1/T_2
- At $\beta = 180^o$ signal is 0.
- At $\beta = 90^\circ, T_1 = T_2, S = M_0/2$

Compare Ellipsoid Signal to Matrix Derivation:

- Matrix (α replaced with β):

$$m(0) = \frac{M_0(1-E_1)}{1+\cos\beta(E_2-E_1)-E_1E_2}\begin{bmatrix}\sin\beta\\0\\E_2+\cos\beta\end{bmatrix}$$

- Approximate $E_{1,2} = e^{-TR/T_{1,2}} \approx 1 - TR/T_{1,2}$ and neglect $TR^2/(T_1T_2)$. Substitute and divide out TR/T_1:

$$M_{xy} = \frac{M_0\sin\beta}{(1+\cos\beta)+(T_1/T_2)(1-\cos\beta)}$$

- Identities $\sin(\beta) = 2\sin(\beta/2)\cos(\beta/2)$, $1+\cos(\beta) = 2\cos^2(\beta/2)$ and $1-\cos(\beta) = 2\sin^2(\beta/2)$:

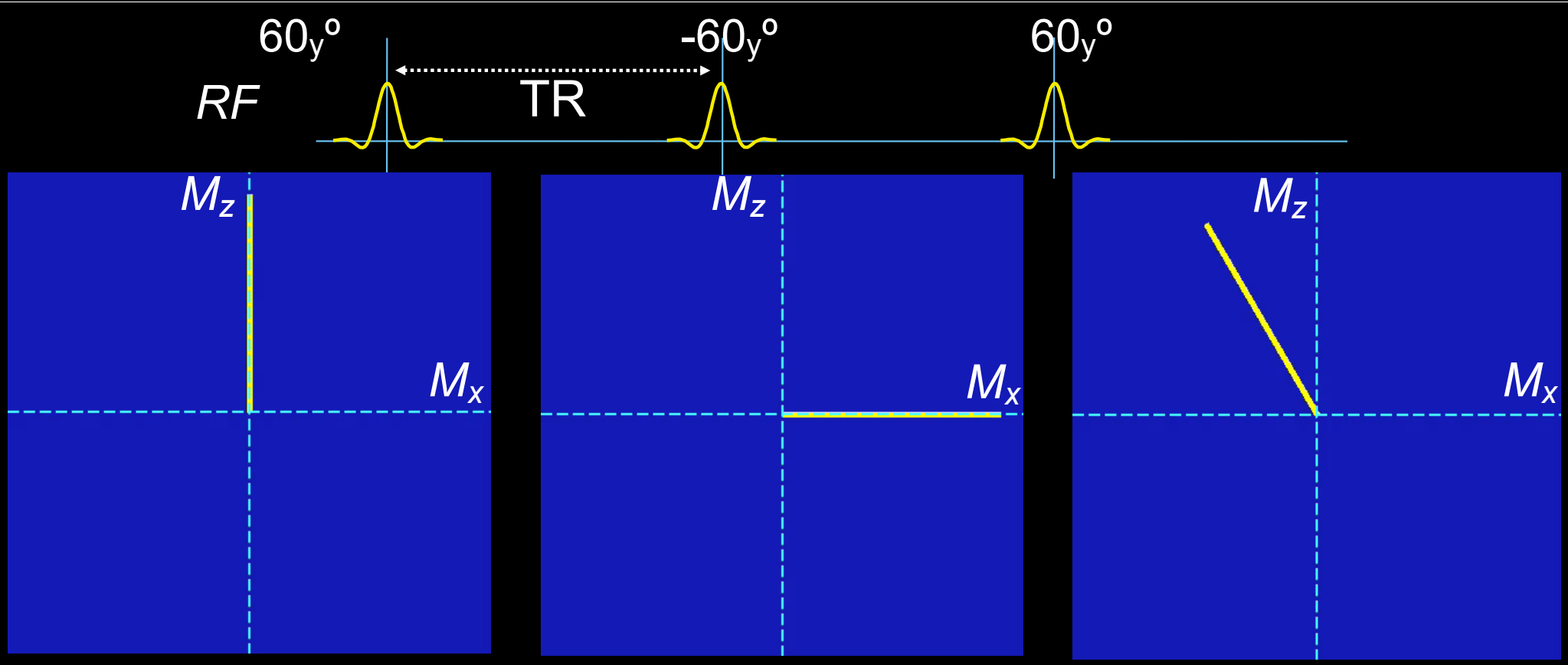
$$M_{xy} = \frac{M_0\sin(\alpha/2)\cos(\alpha/2)}{\cos^2(\alpha/2)+(T_1/T_2)\sin^2(\alpha/2)}$$

- Divide out $\sin(\beta/2)\cos(\beta/2)$:

$$M_{xy} = \frac{M_0}{\cot(\beta/2)+(T_1/T_2)\tan(\beta/2)}$$

Ellipsoidal Derivation!

Balanced SSFP: Transients



- Transient paths differ based on initial state

The transient path depends on the initial condition

Transients: Matrix Solutions

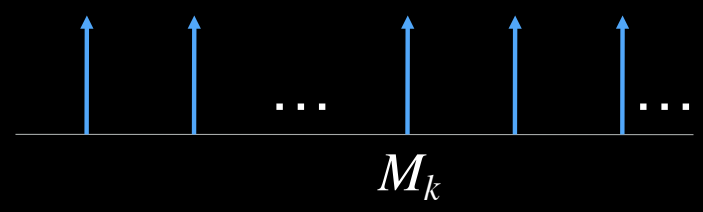
Apply 3x3 matrix scheme:

$$M_{k+1} = A\,M_k + B \qquad [1]$$

In steady-state:

$$M_{ss} = AM_{ss} + B \qquad [2]$$

$$M_{k+1} - M_{ss} = A(M_k - M_{ss}) \qquad [1\text{-}2]$$



Consider Eigenvector Decomposition:

$$A = V\Lambda V^{-1} \qquad M_k - M_{ss} = V\Lambda^k\,[V^{-1}(M_0 - M_{ss})]$$

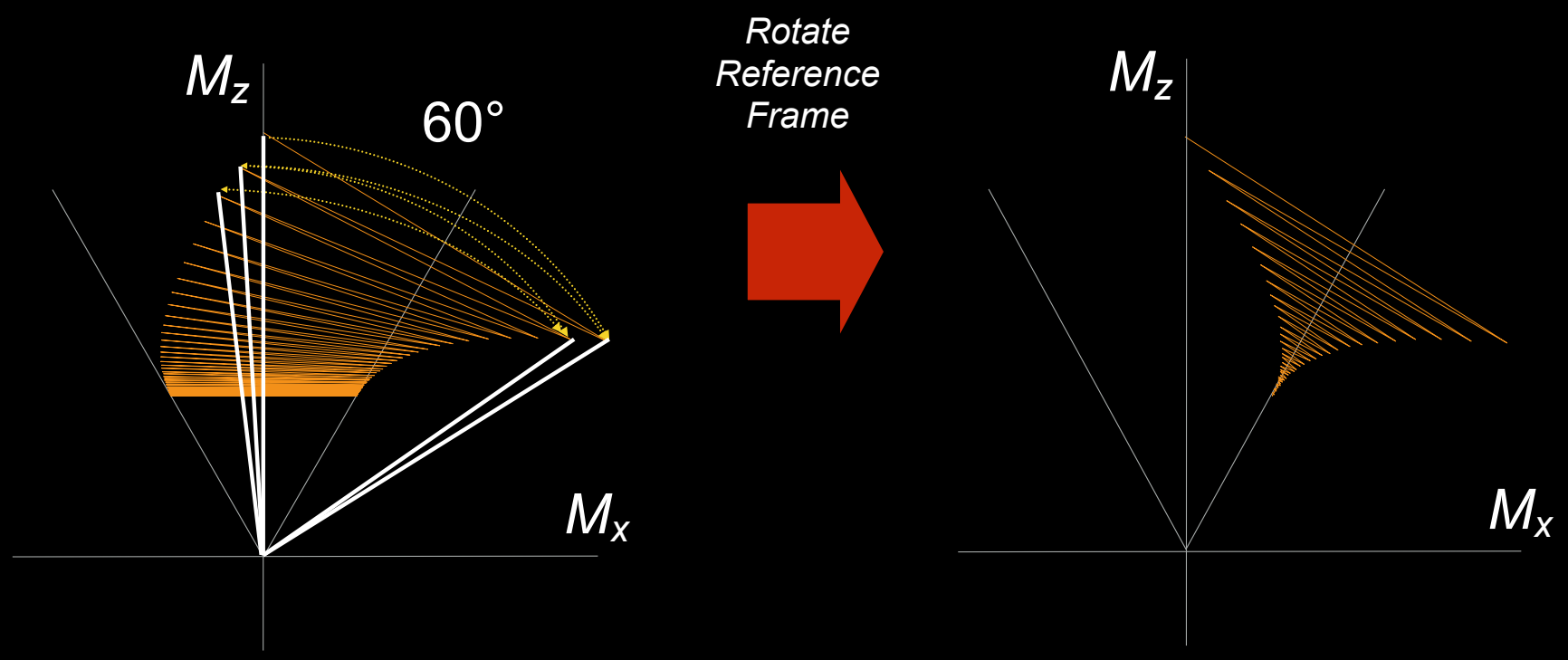
At least one eigenvector/value is real.

Others often oscillatory and die out in steady state

Note in [2] A is mostly rotation, B is small, so M_{ss} lies almost along the real-valued eigenvector

The transient can be characterized by the difference from initial state to steady state. Jaynes 1955

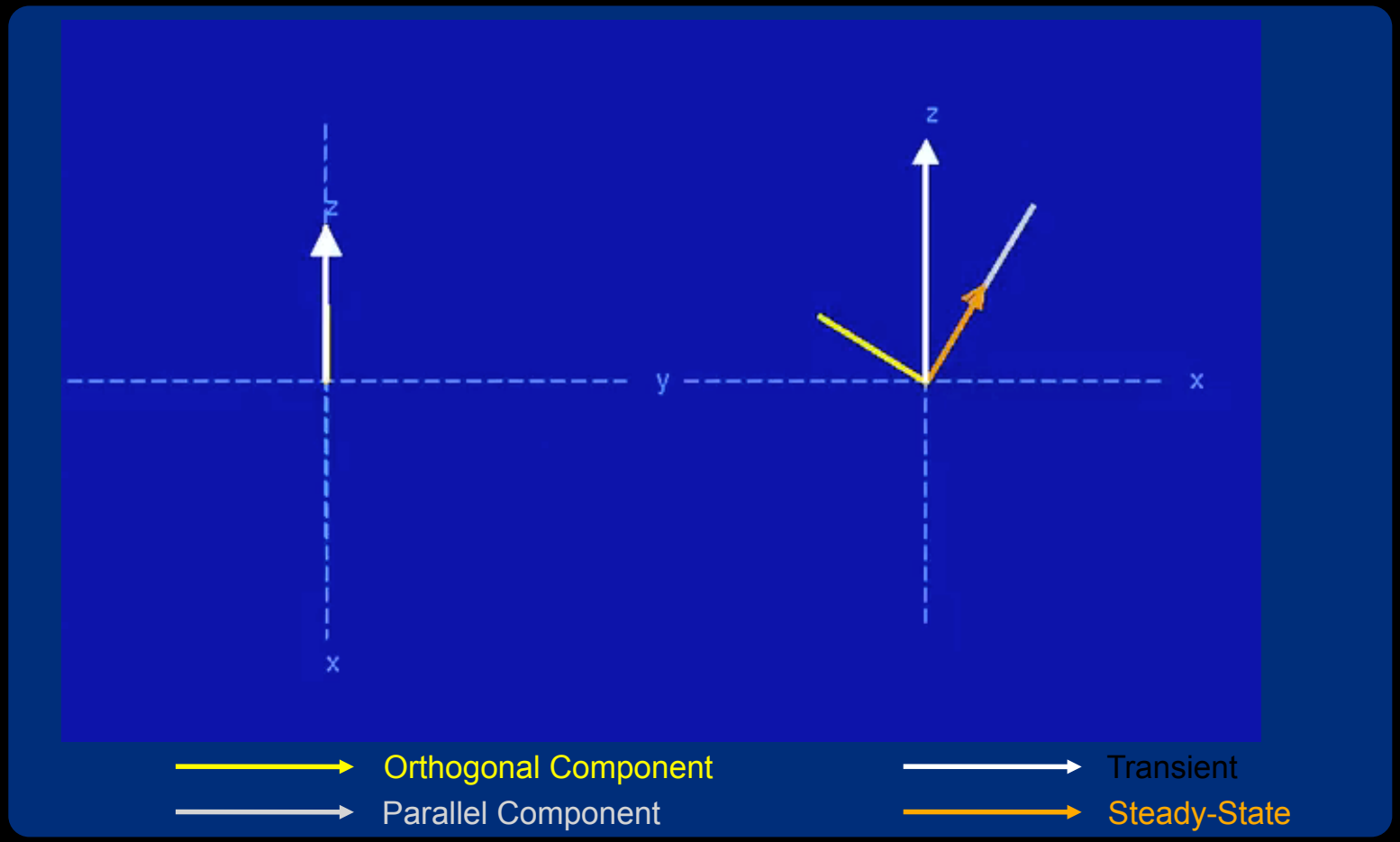
Balanced SSFP: Transient (no precession)



A back-and-forth excitation is the same as a constant rotation with 180° precession

Transient: Off-resonance

Viewed along Steady-State Vector Orthogonal View



The initial component parallel to the steady state decays smoothly, while the orthogonal component oscillates

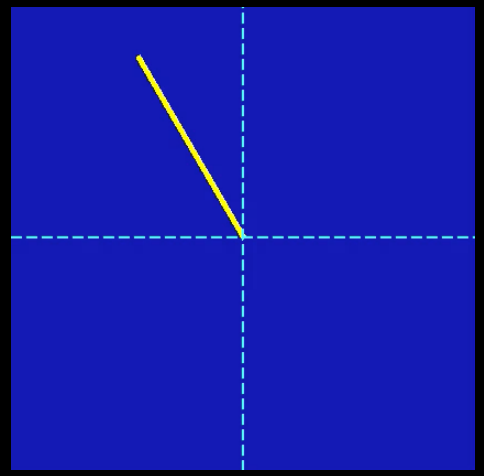
Transients (General)

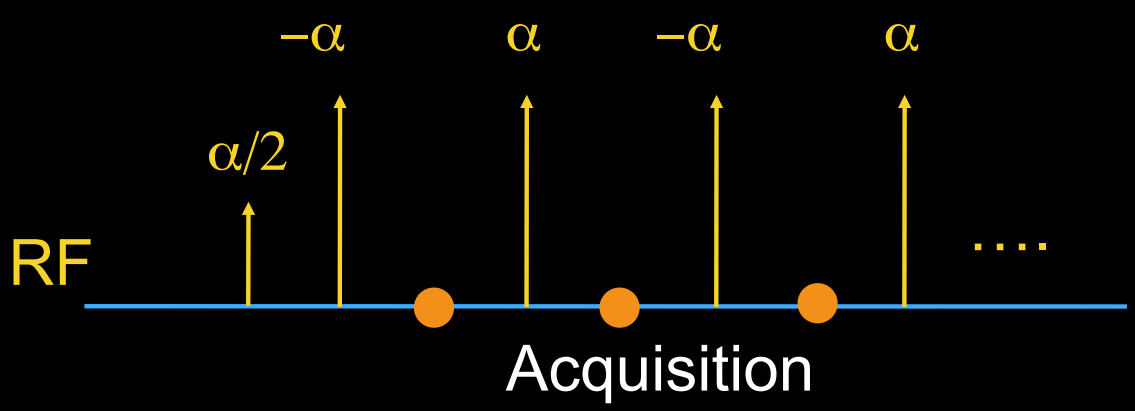
- Generally include two components:
  - Smooth exponential (useful!)
  - Oscillatory (problematic)

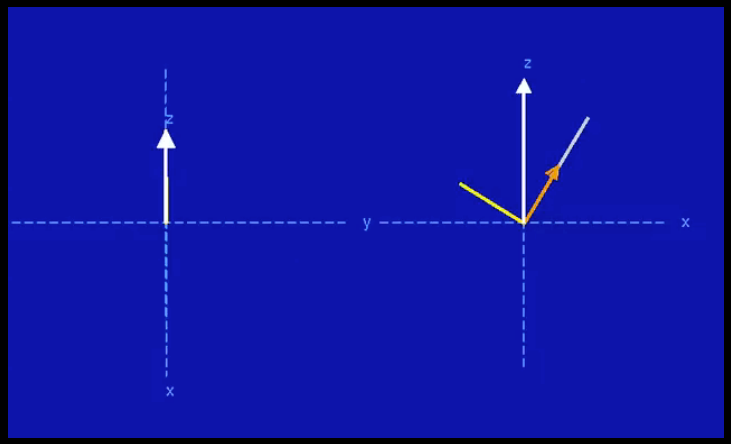
- Smooth transient is along steady-state direction
- Manipulate to steady-state direction to avoid oscillations

Half-TR, α/2 Setup

- First RF pulse has half-amplitude
- Pulse applied TR/2 before next RF pulse
- (More complicated schemes exist)

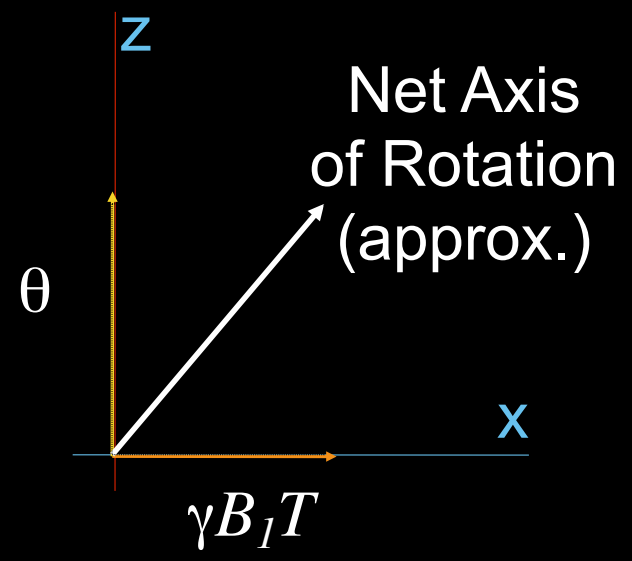


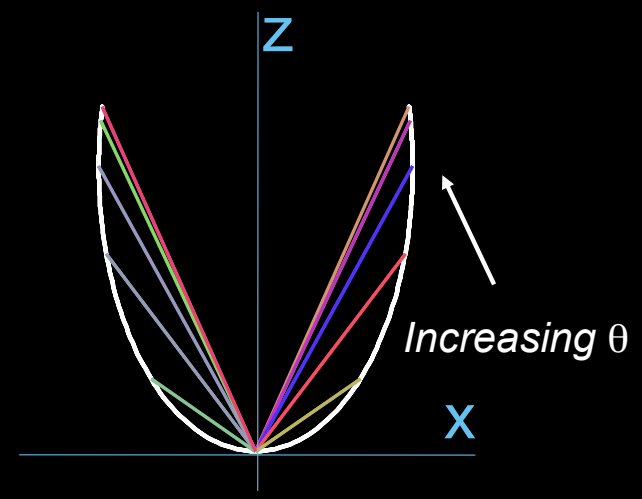




For on-resonance magnetization, places M-M_{ss} along steady-state. Deimling and Heid, 1994

bSSFP Direction: Some Intuition





- Magnetization aligns to rotation axis
- Rotation θ includes RF phase-increment
- As θ approaches 0°, axis is transverse, signal dies out
- Negative θ means steady state on -x

The steady-state lies close to the "effective" rotation axis

Summary

- A matrix derivation of the bSSFP signal in 2D can be done using an alternating reference frame
- A geometric signal derivation arises from the ellipsoid model
- Transients can be described by a matrix exponential:
  - Along steady state, transient is smooth
  - Perpendicular component is oscillatory
- bSSFP steady state can be considered as the “effective” rotation axis

How does this relate to other gradient-echo sequences?

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