STAT 200  HOMEWORK 4  SOLUTION

Remark: Each problem worths 10 points.
Questions? contact with Haiyan Liu.

13  
Sign Test  
The sign test counts the number (denoted by N) of positive observations among $X_1, \ldots, X_{25}$, that is, $N = \sum_{i=1}^{25} I_{X_i > 0}$, where $I_X$ equals 1, if $X > 0$; otherwise, it equals to 0. Under $H_0$: $\mu = 0$, we obtain $N \sim \text{Binomial}(25, 0.5)$. Since $\alpha = 0.05$, we reject $H_0$ when $N \geq 17$.

Under one specific $H_1$: $\mu = 0.3$, we know $N \sim \text{Binomial}(25, p)$, where $p = P(X_1 > 0|X_1 \sim N(0.3, 1)) = 0.62$. Thus we have

$$\text{Power} = \sum_{n=17}^{25} p \left[ N = n | N \sim \text{Binomial}(25, 0.62) \right] \approx 0.35$$

Test Based on Normal Theory  
We reject $H_0$ if $\bar{X} > Z_{0.05} \frac{0.329}{\sqrt{25}} = 0.329$.

So

$$\text{Power} = 1 - \Phi \left( \frac{0.329 - 0.3}{1/5} \right) \approx 0.44$$

14  
Let’s first denote $\Omega = \{ -\infty < \mu < \infty, 0 < \sigma < \infty \}$ and $\omega_0 = \{ \mu = \mu_0, 0 < \sigma < \infty \}$. Now we compute maximum likelihood estimates. Under $\Omega$, $\hat{\mu}^2 = \bar{X}$, and $\hat{\sigma}_1^2 = \frac{\sum(X_i - \bar{X})^2}{n}$; while under $\omega_0$, $\mu$ is known equal to $\mu_0$, and $\hat{\sigma}_0^2 = \frac{\sum(X_i - \mu_0)^2}{n}$. The log ratio of two maximized likelihoods is thus

$$\frac{n}{2} \log \frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2}$$

and the likelihood ratio test rejects for large value of

$$\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} = \frac{\sum(X_i - \mu_0)^2}{\sum(X_i - \bar{X})^2}$$

The alternative expression for the numerator of the ratio is

$$\sum(X_i - \bar{X})^2 + n(\bar{X} - \mu_0)^2$$

equivalently, test rejects for large values of

$$\frac{(\bar{X} - \mu_0)^2}{\sum(X_i - \bar{X})^2}$$
or, for large values of
\[
\frac{|\bar{X} - \mu|}{\sqrt{\sum(X_i - \bar{X})^2}}
\]
equivalent to the $t$ test.

15
The width of the confidence interval is $2Z_{a/2}\sigma\sqrt{\frac{2}{n}}$, so from $2Z_{a/2}\sigma\sqrt{\frac{2}{n}} = 2$ we have
\[
n = 2[Z_{a/2}\sigma]^2 \approx 768
\]

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Power = $P\left[\frac{5 - \frac{F}{\sigma^2}}{\sigma^2} > Z_{a/2}\mu_X - \mu_y = 2\right] = 1 - \phi\left[\frac{Z_{a}\sqrt{\frac{2}{n}} - 2}{\sigma}\right] = 0.5$. Thus
\[
Z_{a}\sigma\sqrt{\frac{2}{n}} - 2 = 0
\]
We obtain $n \approx 82$.

18
Suppose we assign $n_1$ and $n_2$ subjects to two samples respectively, where $n_1 + n_2 = m$. The width of the confidence interval is $2Z_{a/2}\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 2Z_{a/2}\sigma\sqrt{\frac{m}{n_1n_2}}$. In order to provide the shortest C.I., we need to maximize $n_1n_2$ with constraint $n_1 + n_2 = m$, which is achieved when $n_1 = n_2 = m/2$.

Furthermore, Power = $1 - \Phi(Z_{a/2} - \frac{\Delta}{\sigma}\sqrt{\frac{m}{n_1n_2}}) + \Phi(-Z_{a/2} - \frac{\Delta}{\sigma}\sqrt{\frac{m}{n_1n_2}})$. When $\Delta > 0$, the first term is dominant. In order to provide greatest power, we need also to maximize $n_1n_2$; otherwise, the second term is dominant and we need also to maximize $n_1n_2$, which is also achieved by setting $n_1 = n_2 = m/2$.

22
Since $U_Y = \sum_{i=1}^{3} R_{Y_i} - 3 \times 4/2 = R_Y - 6$. Following what is done on P403-404, we achieve the following table

<table>
<thead>
<tr>
<th>$U_Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_Y$</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>Prob</td>
<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

32
For paired design, $\sigma_B^2 = \frac{\sigma_X^2 + \sigma_Y^2 - 2\text{Cov}(X,Y)}{n} = 4$; Power = $1 - \Phi(Z_{a/2} - \frac{\Delta}{2\sigma}) + \Phi(-Z_{a/2} - \frac{\Delta}{2\sigma})$.

For unpaired design, $\sigma_B^2 = \frac{\sigma_X^2 + \sigma_Y^2}{n} = 8$; Power = $1 - \Phi(Z_{a/2} - \frac{\Delta}{2\sigma}) + \Phi(-Z_{a/2} - \frac{\Delta}{2\sigma})$. Use $\alpha = 0.05$; we obtain the plot of power versus $\Delta$. As expected, for the same $\Delta$, paired design has a bigger power than unpaired design in the case with positive correlation.
power versus $\Delta$
34
For paired data, \( \bar{D} = \bar{X} - \bar{Y} = 0.44 \) and \( S_D = \frac{S_p}{\sqrt{n}} = \frac{\frac{463}{15}}{1.96} = 1.196 \).
If the pairing had been erroneously ignored and it had been assumed that the two samples are independent, \( S_D = S_p\sqrt{\frac{2}{n}} = \sqrt{\frac{(n-1)S_p^2 + (n-1)S_q^2}{2n-2}} \sqrt{\frac{2}{n}} = 21.4\sqrt{\frac{2}{15}} = 7.8 \).

We do a paired t-test. \( H_0 \): there is no difference between two methods. \( H_1 \): there is difference. Test statistic
\[
\frac{\bar{D}}{S_D} = 0.44/1.196 = 0.368
\]
It has a p-value of 0.72. So we cannot reject \( H_0 \).

Next we do a signed rank test. there is a zero difference, which we set aside, so \( n = 14 \). The test statistic \( W_+ \) is 61. Under \( H_0 \): \( E(W_+) = \frac{n(n+1)}{4} = 52.5 \) and \( Var(W_+) = \frac{n(n+1)(2n+1)}{24} = 253.75 \). Using normal approximation, we have
\[
Z = \frac{W_+ - E(W_+)}{\sqrt{Var(W_+)}} = \frac{61 - 52.5}{\sqrt{253.75}} = 0.53
\]
It has a p-value of 0.60. We also cannot reject \( H_0 \).

35
See solution on Page A38. If we want to do t-tests, we need to do a paired t-test for part(a) and a two-sample t-test for part(b).

A
We first get \( \hat{R} = \log \left[ \frac{\hat{p}_1(1-\hat{p}_2)}{1-\hat{p}_1}\right] \), where \( \hat{p}_1 = \frac{x_1}{n_1} \) and \( \hat{p}_2 = \frac{x_2}{n_2} \). We write \( f(\hat{p}_1, \hat{p}_2) = \log(\hat{R}) \) and carry out the first-order Taylor expansion for \( f(\hat{p}_1, \hat{p}_2) \) around \( f(p_1, p_2) = \log(R) \) as follows
\[
f(\hat{p}_1, \hat{p}_2) = \log(R) + (\hat{p}_1 - p_1) \frac{\partial f}{\partial \hat{p}_1}\bigg|_{\hat{p}_1=p_1} + (\hat{p}_2 - p_2) \frac{\partial f}{\partial \hat{p}_2}\bigg|_{\hat{p}_2=p_2}
\]
\[
= \log(R) + \frac{\hat{p}_1 - p_1}{p_1(1 - p_1)} + \frac{\hat{p}_2 - p_2}{p_2(1 - p_2)}
\]
When sample size is large, \( \hat{p}_1 \) and \( \hat{p}_2 \) are normally distributed with parameters \( (p_1, \frac{\hat{p}_1(1-\hat{p}_1)}{n_1}) \) and \( (p_2, \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}) \) respectively. So
\[
\log(\hat{R}) \sim N\left( \log(R), \frac{1}{n_1p_1(1-p_1)} + \frac{1}{n_2p_2(1-p_2)} \right)
\]
For the clinical trial example, we have the following confidence interval for \( \log(R) \)
\[
\log(\hat{R}) \pm Z_{\alpha/2} \sqrt{\frac{1}{n_1\hat{p}_1(1-\hat{p}_1)} + \frac{1}{n_2\hat{p}_2(1-\hat{p}_2)}}
\]
Here \( \hat{p}_1 = 0.053, \hat{p}_2 = 0.075 \), \( \log(\hat{R}) = -0.378 \) and \( \sqrt{\frac{1}{n_1\hat{p}_1(1-\hat{p}_1)} + \frac{1}{n_2\hat{p}_2(1-\hat{p}_2)}} = 0.0104 \). Then 95% C.I. for \( \log(R) \) is (-0.58, -0.18); thus for \( R \) is (exp(-0.58), exp(-0.18)).