Stanford Stats 116 Midterm Examination

Closed book and closed notes
Duration: 55 minutes

Spring 2022

Name: ______________________________

Student ID Number: ______________________________

By taking this exam, you agree to be bound by the Stanford Honor Code, meaning specifically in this context that you

- will not give or receive aid in examinations;
- will do your share and take an active part in seeing to it that others as well as yourself uphold the spirit and letter of the Honor Code.

Signature: ______________________________

Question M.1 (A combinatorial identity (10 pts))**:**

(a) (5 pts) Let $n \geq k \geq i \geq 0$ be integers. Show the identity

$$\binom{n}{i} \cdot \binom{n-i}{k-i} = \binom{n}{k} \cdot \binom{k}{i}.$$

(b) (5 pts) Describe in words why the preceding identity holds.

Question M.2 (Counting bridge hands (17pts))**:** In the game of bridge, there are four players, identified by North, East, South, and West (N/E/S/W), with teams (or partnerships) of North/South and East/West. Each player is dealt a hand of 13 cards (of 52 total cards) at random. The deck of cards is a standard deck: there are 13 spades, 13 clubs, 13 hearts, 13 diamonds, and the spades and clubs are black while the diamonds and hearts are red.

(a) (5 pts) How many ways (i.e., how many collections of the 26 cards dealt to N/S) are there for the N/S partnership to have exactly 20 red cards?

(b) (5 pts) How many ways are there for N/S to have at least 20 red cards in the partnership?

(c) (5 pts) How many total ways (i.e., how many collections of 26 cards) can the N/S partnership have?

(d) (2 pts) What is the probability that N/S has at least 20 red cards? (You do not need to simplify).

Question M.3 (Independence (14 pts))**:** Suppose we flip three (fair) coins independently, denoting their outcomes by X_1, X_2, X_3, each either H (heads) or T (tails). Define the events

$$A = \{X_1 = \mathrm{H}\}, \quad B = \{X_2 = \mathrm{H}\}, \quad C = \{X_3 = \mathrm{H}\},$$

and

$$D = \{\text{outcome is one of HHH}, \mathrm{TTH}, \mathrm{THT}, \mathrm{HTT}\}.$$

(Here D represents the outcome of the three coins, so $D = \mathrm{TTH}$ means that $X_1 = \mathrm{T}$, $X_2 = \mathrm{T}$, and $X_3 = \mathrm{H}$, and so on.)

(a) (5 pts) Show that $A \perp\!\!\!\perp D$.

(b) (5 pts) Show that the pair $(A \cap B) \perp\!\!\!\perp D$.

Note that we similarly have that A, B, C are independent, and that by symmetry $B \perp\!\!\!\perp D$, $C \perp\!\!\!\perp D$, and $(A \cap C) \perp\!\!\!\perp D$ and $(B \cap C) \perp\!\!\!\perp D$.

(c) (4 pts) Show that in spite of parts (a) and (b), the four events are not independent.

Question M.4 (Expectations and conditioning (16 pts))**:** Suppose that $X \sim \mathsf{Geom}(p)$, that is, that X has p.m.f. $p_X(i) = P(X = i) = (1-p)^{i-1}p$ for $i = 1, 2, \ldots$. Here we will investigate how much "memory" X maintains about itself.

(a) (6 pts) Give $P(X \geq i)$.

(b) (5 pts) Give

$$P(X = k + i \mid X \geq i).$$

What is the relationship of this quantity to $P(X = k)$?

(c) (5 pts) Suppose your iPod has n songs on it, one of which is Bruce Springsteen's "Born in the USA." You play your iPod on shuffle (where each time a new song begins, the iPod chooses one uniformly at random from your library of n songs) to enjoy some variety. Given that you have listened to $2n$ songs without hearing any songs by the Boss[1], how many additional songs do you expect to listen to before you hear "Born in the USA"? *Hint:* Recall that if $X \sim \mathsf{Geom}(p)$, then $\mathbb{E}[X] = 1/p$, and that the conditional expectation $\mathbb{E}[X \mid A] = \sum_x xP(X = x \mid A)$.

[1] that's Bruce Springsteen

Question M.5* (Extra credit: sampling without replacement (4 extra pts))**:** Suppose that your iPod has n songs on it, one of which remains "Born in the USA," and you listen to them on shuffle, but your iPod has an iVariety mode, where it samples songs *without* replacement. (So in iVariety mode, by the nth song, you've heard each song.) Let X be the first time you hear the song "Born in the USA." Give $\mathbb{E}[X]$.

Scratch paper

Scratch paper