# Stats 116 Problem Set 2

Due: Wednesday, April 13, 5:00 p.m. on Gradescope

Please show your work for each exercise. If you collaborate with someone else—this is fine—be sure to note that in your homework submission. You must each write up separate answer sets. Questions are either from Ross's *A First Course in Probability* or our are home-cooked.

**Question 2.1** (Ross problem 2.29)**:** An urn contains $n$ white and $m$ black balls, where $n$ and $m$ are positive numbers.

(a) If two balls are randomly withdrawn, what is the probability that they are the same color?

(b) If a ball is randomly withdrawn and then replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color?

(c) Show that the probability in part (b) is always larger than the one in part (a).

**Question 2.2** (Ross theoretical exercise 2.13)**:** Prove that $P(E, F^c) = P(E) - P(E, F)$.

**Question 2.3** (Ross theoretical exercise 2.19)**:** An urn contains $n$ red and $m$ blue balls. They are withdrawn one at a time until a total of $r$, where $r \leq n$, red balls have been withdrawn. Find the probability that a total of $k$ balls are withdrawn. *Hint.* A total of $k$ balls will be withdrawn if there are $r - 1$ red balls in the first $k - 1$ withdrawals and the $k$th withdrawal is a red ball.

**Question 2.4** (Ross problem 3.31)**:** There are 15 tennis balls in a box, of which 9 have not previously been used. Three of the balls are randomly chosen, played with, and then returned to the box. Later, another 3 balls are randomly chosen from the box. Find the probability that none of these final 3 balls has ever been used. (You do not need to compute the exact value, but you should write out the formula. If you wish to double-check your formula, it should be about .082.)

**Question 2.5** (Ross problem 3.35)**:** On rainy days, Joe is late to work with probability .3; on nonrainy days, he is late with probability .1. With probability .7, it will rain tomorrow.

(a) Find the probability that Joe is early tomorrow.

(b) Given that Joe was early, what is the conditional probability that it rained?

**Question 2.6** (Ross theoretical exercise 3.5)**:**

(a) Prove that if $E$ and $F$ are mutually exclusive, then

$$P(E \mid E \cup F) = \frac{P(E)}{P(E) + P(F)}$$

(b) Prove that if $E_i$, $i \geq 1$, are all mutually exclusive, then for any $j$,

$$P\left(E_j \mid \bigcup_{i=1}^{\infty} E_i\right) = \frac{P(E_j)}{\sum_{i=1}^{\infty} P(E_i)}.$$

**Question 2.7** (Ross theoretical exercise 3.6)**:** Prove that if $E_1, E_2, \ldots, E_n$ are independent events, then

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = 1 - \prod_{i=1}^{n} \left[1 - P(E_i)\right].$$

*Hint.* Don't use inclusion/exclusion. You may use that if $E_1, \ldots, E_n$ are independent, then so are $E_1^c, \ldots, E_n^c$.

**Question 2.8:** Consider the following stylized scenario of trying to analyze biodiversity in an ecological area. We model this as follows: let $H_k$ be the event that there are $k$ distinct species in the environment, and we assume that $P(H_k) = \frac{6}{\pi^2}\frac{1}{k^2}$ for $k = 1, 2, 3, \ldots$, where the $\frac{6}{\pi^2}$ term normalizes the probabilities so that

$$\sum_{k=1}^{\infty} P(H_k) = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{6}{\pi^2}\frac{\pi^2}{6} = 1.$$

We assume that when $H_k$ holds, we draw a species uniformly at random from the $k$ species in the environment. Suppose we take $n$ samples (observations) from the environment, where we release a species after observing it (i.e., sampling with replacement).

(a) Let $A_n$ be the event that each of the $n$ observations is the same species. Give

$$P(A_n \mid H_k).$$

(b) Using your answer to part (a) to show that given the first $n$ observations are identical species, the probability that the $(n+1)$st species drawn is identical is

$$P(A_{n+1} \mid A_n) = \frac{\zeta(n+2)}{\zeta(n+1)},$$

where $\zeta$ is the *Riemann Zeta Function*, defined as $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ for all complex numbers $s \in \mathbb{C}$ with real part $\mathrm{Re}(z) > 1$. You certainly *do not* need to know about the zeta function to answer this question.

(c) Show that

$$P(H_1 \mid A_n) = \frac{1}{\zeta(n+1)}$$

and so $P(H_1 \mid A_n) \to 1$ as $n \to \infty$. That is, as you observe more and more organisms from a single species, you become more and more certain of the completely non-diverse environment $H_1$.

**Question 2.9**$^\star$ (Extra credit: conditional probabilities)**:** Let the events $A_1, A_2, \dots, A_n, B$ be independent. In this question, you will work out the details to prove the following theorem:

**Theorem 2.9.1.** *For every set $E$ that can be formed by finitely many set operations involving the $A_i$, we have*

$$P(B \mid E) = P(B).$$

To help with this problem a bit, we formally define the types of sets $E$ we use, adopting some language from set theory: we say a collection $\mathcal{A}$ of subsets of a sample space $S$ is an *algebra* over $S$ if the following three conditions hold:

i. $\mathcal{A}$ contains all complements of sets in it, so that if $E \in \mathcal{A}$ then $E^c \in \mathcal{A}$, i.e., $S \setminus E \in \mathcal{A}$.

ii. The empty set is in $\mathcal{A}$, i.e., $\emptyset \in \mathcal{A}$

iii. The collection $\mathcal{A}$ of sets includes all unions and intersections of sets in it, that is,

$$E \cup F \in \mathcal{A} \text{ whenever } E, F \in \mathcal{A}$$

and

$$E \cap F \in \mathcal{A} \text{ whenever } E, F \in \mathcal{A}.$$

The smallest set $\mathcal{A}$ containing $A_1, \dots, A_n$ is called the *algebra generated by* $A_1, \dots, A_n$.

We prove Theorem 2.9.1 in pieces, first by building up the algebra that the $A_i$ generate, then moving to the theorem proper. The first parts (a)–(d) deal with the set algebra; you may, if you wish, simply skip to part (e) to prove the theorem. Throughout the proof, we will use the abuse of notation that $A^1 = A$ and $A^{-1} = A^c$ for sets $A$.

(a) Suppose the set $E = \cup_{i=1}^l E_i$ where the $E_i$ are not necessarily disjoint. Show that

$$E = \bigcup_{i=1}^{l} G_i$$

for sets $G_i = E_i \setminus \{E_1 \cup \dots E_{i-1}\} = E_1^c \cap E_2^c \cap \dots \cap E_{i-1}^c \cap E_i$, which are disjoint.

(b) Suppose the set $E$ is of the form $E = A_1^{\alpha_1} \cap A_2^{\alpha_2} \cap \dots \cap A_k^{\alpha_k}$, where each $\alpha_j \in \{\pm 1\}$. Show that

$$E^c = \bigcup_{j=1}^{k} E_j$$

where the sets

$$E_j = A_j^{-\alpha_j} \setminus \{A_1^{-\alpha_1} \cup \dots \cup A_{j-1}^{-\alpha_{j-1}}\} = A_j^{-\alpha_j} \cap A_1^{\alpha_1} \cap \dots \cap A_{j-1}^{\alpha_{j-1}}$$

are all disjoint.

(c) Suppose that the sets $E$ and $F$ are of the form $E = \cup_{i=1}^l E_i$ and $F = \cup_{j=1}^m F_j$, where the $E_i$ are all disjoint from one another and the $F_j$ are all disjoint from one another. Argue that

$$E \cap F = \bigcup_{i=1}^{l} \bigcup_{j=1}^{m} G_{ij}$$

for sets of the form $G_{ij} = E_i \cap F_j$, where the $G_{ij}$ are all disjoint.

(d) Suppose the sets $E$ and $F$ are of the form $E = \cup_{i=1}^{l} E_i$ and $F = \cup_{j=1}^{m} F_j$. Define the lexicographic order on pairs $(i, j)$ by $(i', j') \prec (i, j)$ if $i' < i$ or $i' = i$ and $j' < j$. Conclude using part (a) that we may write

$$E \cup F = \bigcup_{i=1}^{l} \bigcup_{j=1}^{m} G_{ij}$$

where $G_{ij}$ are disjoint and equal to the disjoint union

$$G_{ij} = \left( E_i \cap \bigcap_{(i',j') \prec (i,j)} (E_{i'}^c \cap F_{j'}^c) \right) \cup \left( F_j \cap E_i^c \cap \bigcap_{(i',j') \prec (i,j)} (E_{i'}^c \cap F_{j'}^c) \right).$$

Parts (a)–(d) actually shown that for the algebra $\mathcal{A}$ generated by $A_1, \ldots, A_n$, we can write any set $E \in \mathcal{A}$ (or event $E$) as the disjoint union

$$E = \bigcup_{j=1}^{l} E_j \tag{2.1}$$

where each set $E_j$ is an intersection of a finite number of sets $A_i$ and their complements, i.e.

$$E_j = A_{i_1}^{\alpha_1} \cap A_{i_2}^{\alpha_2} \cap \ldots \cap A_{i_k}^{\alpha_k}$$

for some signs $\alpha \in \{\pm 1\}^k$ and indices $i_1, \ldots, i_k \in \{1, \ldots, n\}$, and the $E_j$ are disjoint.

(e) Using the identity (2.1), prove Theorem 2.9.1. You may use the fact that if $A_1, \ldots, A_n, B$ are (mutually) independent, then so are $A_1^{\alpha_1}, \ldots, A_n^{\alpha_n}, B^{\alpha_0}$ for any $\alpha_j \in \{\pm 1\}$.