Stats 116 Problem Set 8

Due: Wednesday, May 25 5:00 p.m. on Gradescope

Please show your work for each exercise. If you collaborate with someone else—this is fine—be sure to note that in your homework submission. You must each write up separate answer sets. Questions are either from Ross's *A First Course in Probability*, Blitzstein and Hwang's *Introduction to Probability, Second Edition*, or our are home-cooked special sauce.

Question 8.1 (Some properties of conditional expectation)**:** Show the following. You may assume that X is discrete or continuous, just make sure you note your choice in your argument.

(a) (2 pts) The "taking out what's known" property of conditional expectation: for any function h,

$$\mathbb{E}[Xh(Y) \mid Y = y] = h(y)\mathbb{E}[X \mid Y = y]$$

(and so, of course, $\mathbb{E}[Xh(Y) \mid Y] = h(Y)\mathbb{E}[X \mid Y]$).

(b) (2 pts) Independence properties: if X and Y are independent, then $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$.

(c) (2 pts) The optimality of conditional expectation as a predictor of X given an observation Y: if h is any function, then

$$\mathbb{E}[(X - h(Y))^2] \geq \mathbb{E}[(X - \mathbb{E}[X \mid Y])^2].$$

Hint: Let $g(y) = \mathbb{E}[X \mid Y = y]$. Expand the square in $(x - h(y))^2 = (x - g(y) + g(y) - h(y))^2$, then use the taking out property of conditional expectation.

Question 8.2 (Blitzstein and Hwang Exercise 8.2, 4 pts)**:** While Fred is sleeping one night, X legitimate emails and Y spam emails are sent to him. Suppose that X and Y are independent, with $X \sim \mathsf{Poisson}(10)$ and $Y \sim \mathsf{Poisson}(40)$. When he wakes up, he observes that he has 30 new emails in his inbox. Given this information, what is the expected value of how many new legitimate emails he has?

Question 8.3 (Blitzstein and Hwang Exercise 8.29, 2 pts)**:** Show that if $\mathbb{E}[X \mid Y] = c$ is constant, then X and Y are uncorrelated.

Question 8.4 (Blitzstein and Hwang Exercise 8.30, 3 pts)**:** Show by example that it is possible to have uncorrelated X and Y (that is, X and Y with $\mathrm{Cov}(X, Y) = 0$) such that $\mathbb{E}[Y \mid X]$ is non-constant. *Hint:* Consider a normal and its square.

Question 8.5: An auction is a *second price auction*, or a *Vickrey auction*, if the winner pays the amount of the second place bid. You will use conditional expectations to give one version of the famous result that bidders are incentivized to bid their true values rather than strategically tweaking their bids. Suppose there is an auction of an item, and the item has a value $v^\star > 0$ to you. There are $n \geq 1$ other bidders whose bids are indepenent of yours (though they may depend on one another's bids). If your bid is larger than or equal to B, the largest bid of the remaining bidders, you win the auction, pay B and receive the item of value $v^\star$. Otherwise, you lose the auction and pay nothing.

Let P be the amount you pay in the auction (so $P = 0$ if you lose) and let V be the value of what you receive in the auction (so $V = 0$ if you lose). Your return in the auction is then the (random) difference $V - P$.

(a) (2 pts) Show that if your bid is b, then

$$\mathbb{E}[V - P \mid \text{Win the auction}] = v^\star - \mathbb{E}[B \mid B \leq b],$$

and show that $\mathbb{E}[V - P \mid \text{Lose the auction}] = 0$ no matter your bid.

(b) (2 pts) Show that if your bid is $v^\star$, then

$$\mathbb{E}[V - P] \geq 0,$$

that is, bidding always has (in expectation) nonnegative value.

(c) (3 pts) Define the expected return function

$$r(b) := \mathbb{E}[V - P \mid \text{your bid is } b].$$

Show that if $b < v^\star$, meaning you bid less than your value, then

$$r(v^\star) - r(b) \geq 0,$$

and also that if $P(b < B < v^\star) > 0$, then $r(v^\star) - r(b) > 0$. That is, bidding less than your (internal) value for the item leaves some expected return on the table.

(d) (2 pts) Show similarly that if $b > v^\star$, then $r(v^\star) - r(b) \geq 0$, and if $P(b > B > v^\star) > 0$, then $r(v^\star) - r(b) > 0$.

(e) (2 pts) In a *first price* auction, the winner pays the winning bid. Let P_1 be the amount you pay in a first-price auction. Show that you have less returns in such an auction, i.e.,

$$\mathbb{E}[V - P_1 \mid \text{Win the auction}] \leq \mathbb{E}[V - P \mid \text{Win the auction}].$$

(f) (1pt) Why do you think auctioneers, e.g., Google's ad services, use first price auctions?