

Stanford Stats 117 Midterm Examination

Closed book and single sheet of paper

Duration: 100 minutes

Spring 2026

Name:¹ _____

Student ID Number: _____

By taking this exam, you agree to be bound by the Stanford Honor Code, meaning specifically in this context that you

- will not give or receive aid in examinations;
- will do your share and take an active part in seeing to it that others as well as yourself uphold the spirit and letter of the Honor Code.

We have included scratch paper at the end of this exam. Please use it for any calculations you need, and do not remove it from the exam.

Signature: _____

¹It might be good to write your name on each page of your exam

Question M.1 (True/Not necessarily): Each of these questions provides a statement involving events and probabilities. Determine whether each *always* holds, in which case it is true; or *may* not hold, in which case the statement is not necessarily true. If the answer is not necessarily true, give one sentence explaining this. For example, the statement “for an event A , $P(A) > 0$ ” is not necessarily true: the set $A = \emptyset$ falsifies it. You do not need to justify true statements. We will give 1 point of extra credit for each statement you identify correctly as being *guaranteed* to be false: there is no way it can be true.

(a) (4 pts) For events A_1, A_2, \dots, A_n , we have

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n).$$

(b) (4 pts) For events A_1, A_2, A_3, \dots , we have

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n).$$

(c) (4 pts) For events A and B , we have

$$P(A \cap B) = P(A)P(B).$$

(d) (4 pts) For events A and B , we have

$$P(A \cup B) > P(A).$$

(e) (4 pts) Let $0 < p < 1$ and $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent random variables, where we take the “Poisson scaling” $p = \frac{\lambda}{n}$. Then $X + Y \sim \text{Binomial}(2n, p)$.

Question M.2: In Texas Hold'em, players are dealt two "hole cards" face down, and then a series of five cards is dealt face up (the "community cards," dealt as the "flop," the "turn," and the "river," though these names are unimportant); assume that the deal is truly uniformly random. You seek the best five-card poker hand from any combination of the seven cards dealt, including your hole cards. Suppose your hole cards consist of a pair (e.g., a two of hearts and a two of spades, or a seven of clubs and a seven of diamonds); this is called a "pocket pair."

(a) **(10 pts)** Given that you have a pocket pair, what is the probability that you achieve four of a kind using that pair?

(b) **(10 pts)** Given that you have a pocket pair, what is the probability that you achieve four of a kind at all?

Question M.3 (Energy levels): In physics, a system takes on a random state value $S \in \mathbb{N} = \{0, 1, 2, \dots\}$, where for a state s the associated energy level is $E(s) := s + \frac{1}{2}$. For an inverse temperature $\beta > 0$, the probability that the realized state $S = s$ is

$$P(S = s) = \frac{1}{Z(\beta)} e^{-\beta E(s)}, \quad s = 0, 1, \dots, \quad \text{where } Z(\beta) = \sum_{s=0}^{\infty} e^{-\beta E(s)} = \sum_{s=0}^{\infty} e^{-\beta(s+\frac{1}{2})}.$$

Though this is not important, we call $Z(\beta)$ the *partition function*.

(a) **(6 points)** Give a closed form for $Z(\beta)$ as a function of β .

(b) **(13 pts)** For a given inverse temperature β , what is the expected energy level $\mathbb{E}[E(S)]$? *Hint.* Recall that if $X \sim \text{Geom}(p)$, then $\mathbb{E}[X] = \frac{1}{p}$. To check your answer, you may use the identity $\frac{\partial}{\partial \beta} \log Z(\beta) = -\mathbb{E}[E(S)]$; if you use this directly to obtain $\mathbb{E}[E(S)]$, you must prove it.

- (c) **(1 pts)** If T is the temperature of the system (so $\beta = 1/T$), then what is the limit of the expected energy $\mathbb{E}[E(S)]$ of the system as $T \uparrow \infty$, that is, the temperature increases to ∞ ?

Question M.4 (Generating a random integer): We would like to generate a random integer R taking each value in $\{0, 1, 2, \dots, 999\}$ with equal probability. Suppose we have a fair 10-sided die with faces numbered $\{0, 1, 2, \dots, 9\}$. Suppose we roll the die 3 times, and let D_1, D_2, D_3 be the outcome of the independent rolls. Now define the random number

$$R = 10^2 \cdot D_3 + 10^1 \cdot D_2 + 10^0 \cdot D_1 = 100 \cdot D_3 + 10 \cdot D_2 + D_1.$$

Fix an integer $k \in \{0, \dots, 999\}$ with digits $k = d_3d_2d_1$ (so $d_3 \in \{0, \dots, 9\}$ is the 100s place digit, d_2 the 10s place, d_1 the 1s place).

(a) **(10 pts)** Show the following equality of events:

$$\{R = k\} = \{D_1 = d_1\} \cap \{D_2 = d_2\} \cap \{D_3 = d_3\}.$$

(b) **(10 pts)** Show that $P(R = k) = \frac{1}{1000}$, that is, R is equally likely to take any of the values in $\{0, 1, 2, \dots, 999\}$.

Question M.5: Geiger, excited about his radiation counting invention and eager to monetize it, begins a casino based on the following game: an individual counts the number Y of radiation events over a T second window, so for a known background radiation rate $\lambda > 0$ we have $Y \sim \text{Poisson}(\lambda T)$. At the end of the time period T , the individual receives a payout of 2^Y dollars, that is, they begin with a wealth of \$1 and double it for each radiation event.

(a) **(12 pts)** Geiger chooses the length of time T to be $T = \frac{1}{2\lambda}$. Give the expected payout $\mathbb{E}[2^Y]$.

(b) **(3 pts)** If Geiger wishes to make money (on average), how much should he charge patrons to play his game (when he chooses $T = \frac{1}{2\lambda}$ as in part (a))?

Question M.6* (Extra Credit, 6 pts): We demonstrate how to simulate an event with probability p for any $0 \leq p \leq 1$ from a 10-sided die. Let $D_i \in \{0, 1, 2, \dots, 9\}$ be independent random variables with $P(D_i = 0) = P(D_i = 1) = \dots = P(D_i = 9) = \frac{1}{10}$. Define

$$R = \sum_{i=1}^{\infty} 10^{-i} \cdot D_i = 0.D_1D_2D_3D_4\dots$$

to be the real number with decimal expansion whose i th entry is D_i .

(a) (2 pts) Let $0 < p < 1$ have decimal expansion $p = 0.d_1d_2d_3\dots$, so that d_i is its i th digit. For each $n \in \{1, 2, \dots\}$, define the event

$$E_n := \{D_n < d_n\} \cap \{D_1 = d_1, D_2 = d_2, \dots, D_{n-1} = d_{n-1}\}.$$

Show that the events E_n are all disjoint and

$$\{R < p\} = \bigcup_{n=1}^{\infty} E_n.$$

(b) (1 pt) Show that for any n and any sequence of digits $d_i \in \{0, 1, \dots, 9\}$,

$$P(E_n) = P(D_1 = d_1, \dots, D_{n-1} = d_{n-1}, D_n < d_n) = 10^{-n} \cdot d_n.$$

(c) **(1 pt)** Show that $P(R < p) = p$ for any $0 < p < 1$.

(d) **(2 pts)** Give a procedure for actually generating an event with probability exactly $p \in (0, 1)$ using 10-sided dice. On average, how many dice rolls does your procedure take?

Scratch page.

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