Stats 117 Problem Set 2

Due: Monday, April 13 5:00 p.m. on Gradescope

Please show your work for each exercise. If you collaborate with someone else—this is fine—be sure to note that in your homework submission. You must each write up separate answer sets. Any starred exercise is optional: they are extra challenging theoretical exercises for further developing your mastery.

Question 2.1: In a given population, an individual has a walnut allergy with probability .05. The skin prick test for allergies has a very low false negative rate: upon getting their skin pricked, an individual who is allergic to walnuts will react to the test with probability .99. On the other hand, its false positive rate is high: an individual without a walnut allergy will react to the test with probability .5. An individual is given the skin prick test for walnut allergies and their response is measured.

(a) What is the probability that the individual tests positive for a walnut allergy?

(b) Given that the individual tested positive for a walnut allergy, what is the probability that the individual is actually allergic to walnuts?

Question 2.2 (Some set theory questions)**:** Let $A_1, A_2, A_3, \ldots$ be sets, and B be a set.

(a) If each pair of sets $A_i \cap A_{i+1}$ has non-empty intersection, that is, $A_i \cap A_{i+1} \neq \emptyset$, is

$$\bigcap_{i=1}^{\infty} A_i$$

non-empty? If so, demonstrate it; if not, give a counterexample.

(b) Show the distributive law that for any sets A_i and B,

$$\left(\bigcap_{i=1}^{\infty} A_i\right) \cup B = \bigcap_{i=1}^{\infty} (A_i \cup B).$$

(c) *Extra hard:* If a collection of sets A_i satisfies the *finite intersection property* that for any finite collection $A_1, \ldots, A_n$ we have $A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset$, are we guaranteed that $\cap_{i=1}^{\infty} A_i$ is non-empty?

Hint. To show that two sets E and F are equal, it is enough to show that $E \subseteq F$ and $F \subseteq E$, that is, if $x \in E$, then $x \in F$ (which is equivalent to $E \subseteq F$), and conversely, if $x \in F$, then $x \in E$ (which is equivalent to $F \subseteq E$).

Question 2.3 (The law of total probability)**:** Let Ω be a sample space, and let $E \subset \Omega$ be an event and let $A_i \subset \Omega$ be events, not necessarily disjoint, with $\cup_{i \geq 1} A_i = \Omega$. Show that

$$P(E) = P\left(E \cap \bigcup_{i \geq 1} A_i\right) \leq \sum_{i \geq 1} P(E \cap A_i).$$

Question 2.4 (*Art of Chance* 4.7, extended)**:** Let Ω be the collection of all infinite sequences of coint tosses.

(a) Let the probability function P as defined in Example 3.11 of the book, so that samples are independent. Show that

$$P(\text{infinitely many heads}) = 1.$$

Hint. Write the event in terms of unions and intersections of events $A_i = \{\omega \mid \omega_i \text{ is heads}\}$, then consider the complement.

(b) If the flips are *not* necessarily independent, does this result remain true?

Question 2.5 (*Art of Chance* 5.2, variant)**:** A box has 10 apples, 5 oranges, and 1 mango. Consider the process of choosing a fruit from the box at random (so, e.g., with the first 16 fruits, the probability of a mango is 1/16), upon which all fruits of the same type are removed from the box, and the procedure repeats.

(a) Calcualte the probability that, given three fruit choices, the order of fruit choices is Mango, Orange, Apple.

(b) Calculate the probability that Mango and Oranges will be the first two picks of fruit. Do different orderings of the same three fruits have the same probability?

Question 2.6* (Some non-intuitive problems that arise for finitely additive probabilities)**:** A collection of sets $\mathcal{A}$ is called a *set algebra* if it is closed under finite set operations: that is, if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$, and if A and B belong to $\mathcal{A}$, then $A \cup B \in \mathcal{A}$ and $A \cap B \in \mathcal{A}$.

(a) Let $\mathcal{A}$ be a set algebra. Show that if $A_1, \ldots, A_n \in \mathcal{A}$, then $A_1 \cup \cdots \cup A_n \in \mathcal{A}$, and also $A_1 \cap \cdots \cap A_n \in \mathcal{A}$. (That is, $\mathcal{A}$ is closed under *finite* set operations.)

Now let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the natural numbers. Let $\mathcal{F}$ be the collection of finite or co-finite subsets of $\mathbb{N}$, that is, the collection of sets $A \subset \mathbb{N}$ for which A or A^c is finite.

(b) Show that $\mathcal{F}$ is an algebra. In particular, show that if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$ and $A \cap B \in \mathcal{F}$. Show also that $\mathbb{N} \in \mathcal{F}$.

(c) Consider the probability distribution on the natural numbers $\Omega = \mathbb{N}$, specifically, on sets $A \in \mathcal{F}$ defined by

$$P(A) = \begin{cases} 1 & \text{if } A^c \text{ is finite} \\ 0 & \text{if } A \text{ is finite.} \end{cases}$$

Show that P satisfies the *finite* axioms of probability on $\mathcal{F}$:

(i) Non-negativity: $P(A) \geq 0$ for all $A \in \mathcal{F}$

(ii) Normalization: $P(\Omega) = 1$

(iii) Finite additivity: if $A_1, \ldots, A_n \in \mathcal{F}$ are disjoint, then $P(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^n P(A_i)$.

Hint. If $A_i \in \mathcal{F}$ is infinite, then A_i^c is finite. Must $(A_1 \cup \cdots \cup A_n)^c$ be finite?

(d) Now, show that $P(\{n\}) = 0$ for each natural number $n \in \mathbb{N}$, but $P(\mathbb{N}) = 1$, that is, the probability that *some* natural number results is 1, but each natural number has probability 0.