Stats 117 Problem Set 2

Due: Monday, April 13 5:00 p.m. on Gradescope

Please show your work for each exercise. If you collaborate with someone else—this is fine—be sure to note that in your homework submission. You must each write up separate answer sets. Any starred exercise is optional: they are extra challenging theoretical exercises for further developing your mastery.

Question 2.1: On rainy days, Joe is late to work with probability .3; on nonrainy days, he is late with probability .1. With probability .7, it will rain tomorrow.

(a) Find the probability that Joe is early tomorrow.

(b) Given that Joe was early, what is the conditional probability that it rained?

Question 2.2 (Some set theory questions)**:** Let $A_1, A_2, A_3, \ldots$ be sets, and B be a set.

(a) Show that if the A_i are disjoint, then the sets

$$A_i \cap B$$

are also all disjoint.

(b) Show the distributive law that

$$\left(\bigcup_{i=1}^{\infty} A_i\right) \cap B = \bigcup_{i=1}^{\infty}(A_i \cap B).$$

Hint. To show that two sets E and F are equal, it is enough to show that $E \subseteq F$ and $F \subseteq E$, that is, if $x \in E$, then $x \in F$ (which is equivalent to $E \subseteq F$), and conversely, if $x \in F$, then $x \in E$ (which is equivalent to $F \subseteq E$).

Question 2.3 (The law of total probability)**:** Let Ω be a sample space, and let $E \subset \Omega$ be an event and let $A_i \subset \Omega$ be disjoint events with $\cup_{i \geq 1} A_i = \Omega$ (sometimes we call this a *partition of the sample space* Ω). Show that

$$P(E) = P\left(E \cap \bigcup_{i \geq 1} A_i\right) = \sum_{i \geq 1} P(E \cap A_i).$$

Question 2.4 (*Art of Chance* 4.7)**:** Let Ω be the collection of all infinite sequences of coint tosses, with the probability function P as defined in Example 3.11 of the book. Show that

$$P(\text{infinitely many heads}) = 1.$$

Hint. Write the event in terms of unions and intersections of events $A_i = \{\omega \mid \omega_i \text{ is heads}\}$, then consider the complement.

Question 2.5 (*Art of Chance* 5.2)**:** Continuing Example 5.3, the teams that won the first three picks in the 1992 NBA draft were the Orlando Magic, Charlotte Hornets, and the Minnesota Timberwolves.

(a) Calculate the probability that these three teams would win the first three picks, in this order.

(b) Calculate the probability that these three teams would win the first three picks, in *any* order. Do different orderings of the same three teams have the same probability?

Question 2.6* (Some non-intuitive problems that arise for finitely additive probabilities)**:** A collection of sets $\mathcal{A}$ is called a *set algebra* if it is closed under finite set operations: that is, if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$, and if A and B belong to $\mathcal{A}$, then $A \cup B \in \mathcal{A}$ and $A \cap B \in \mathcal{A}$.

(a) Let $\mathcal{A}$ be a set algebra. Show that if $A_1, \ldots, A_n \in \mathcal{A}$, then $A_1 \cup \cdots \cup A_n \in \mathcal{A}$, and also $A_1 \cap \cdots \cap A_n \in \mathcal{A}$. (That is, $\mathcal{A}$ is closed under *finite* set operations.)

Now let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the natural numbers. Let $\mathcal{F}$ be the collection of finite or co-finite subsets of $\mathbb{N}$, that is, the collection of sets $A \subset \mathbb{N}$ for which A or A^c is finite.

(b) Show that $\mathcal{F}$ is an algebra. In particular, show that if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$ and $A \cap B \in \mathcal{F}$. Show also that $\mathbb{N} \in \mathcal{F}$.

(c) Consider the probability distribution on the natural numbers $\Omega = \mathbb{N}$, specifically, on sets $A \in \mathcal{F}$ defined by

$$P(A) = \begin{cases} 1 & \text{if } A^c \text{ is finite} \\ 0 & \text{if } A \text{ is finite.} \end{cases}$$

Show that P satisfies the *finite* axioms of probability on $\mathcal{F}$:

(i) Non-negativity: $P(A) \geq 0$ for all $A \in \mathcal{F}$

(ii) Normalization: $P(\Omega) = 1$

(iii) Finite additivity: if $A_1, \ldots, A_n \in \mathcal{F}$ are disjoint, then $P(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^n P(A_i)$.

Hint. If $A_i \in \mathcal{F}$ is infinite, then A_i^c is finite. Must $(A_1 \cup \cdots \cup A_n)^c$ be finite?

(d) Now, show that $P(\{n\}) = 0$ for each natural number $n \in \mathbb{N}$, but $P(\mathbb{N}) = 1$, that is, the probability that *some* natural number results is 1, but each natural number has probability 0.