# Stats 117 Problem Set 3

Due: Monday, April 20 5:00 p.m. on Gradescope

Please show your work for each exercise. If you collaborate with someone else—this is fine—be sure to note that in your homework submission. You must each write up separate answer sets. Any starred exercise is optional: they are extra challenging theoretical exercises for further developing your mastery.

**Question 3.1:** There are 4 coins in a box. Two are two-headed coins, one is a fair coin, and the fourth is a weighted coin that comes up heads 75% of the time. When one of the 4 coins is selected at random and flipped, it comes up heads. What is the probability that it was a two-headed coin?

**Question 3.2:** A player shooting free-throws gets "in a rhythm", so that each time they make a shot, the probability they make the next shot is halfway between 1 and the probability they made the first shot:

$$\mathbb{P}(\text{make next shot} \mid \text{made previous shot}) = \frac{1}{2} + \frac{1}{2} \cdot \mathbb{P}(\text{made previous shot}).$$

On the other hand, each time they miss, the probability of making the next shot halves:

$$\mathbb{P}(\text{make next shot} \mid \text{missed previous shot}) = \frac{1}{2}\mathbb{P}(\text{made previous shot}).$$

The player takes two free throws. Assuming that $\mathbb{P}(\text{makes first}) = \frac{1}{2}$, give the p.m.f. of $Y$, the number of points the player scores.

**Question 3.3** (Packets and queues)**:** In a network system, packets are sent through a router, which builds a queue of individual packets to be sent on to a destination. At each time step, the queue can either discharge a packet, sending it on, or receive a new packet, increasing the number of packets being stored. Imagine the queue has a total capacity of $m$ packets, and at each time step, with probability $\frac{1}{2}$ a new packet arrives before a packet is sent on, meaning the total number $x$ of packets in the queue increments by one, and with probability $\frac{1}{2}$ a packet is discharged, so the total number of packets decrements by 1. We remove the router from service if its queue is empty, because we do not need to use it; if the router overflows—meaning it hits capacity $m$—we must re-provision (and purchase a new router).

Define the probability

$$p_i := \mathbb{P}(\text{the router overflows} \mid \text{current queue length is } i).$$

(a) Justify the equality that for $i = 1, 2, \ldots, m-1$,

$$p_i = \frac{1}{2}(p_{i+1} + p_{i-1}) \text{ and so } p_{i+1} - p_i = p_i - p_{i-1}.$$

That is, the probability the router overflows given that we begin with $i$ packets in the queue is half $\times$ the probability it overflows beginning from $i+1$ packets plus half $\times$ the probability it overflows from $i-1$.

(b) Why do $p_0 = 0$ and $p_m = 1$?

(c) Show that $p_i = \frac{i}{m}$ for each $i = 0, 1, \ldots, m$. (It is enough to verify that this solves the equation in part (a).)

Convince yourself that if we overprovision sufficiently, meaning the capacity $m \to \infty$, then we will remove the router from service before it overflows (that is, $p_i \to 0$ for any fixed initial wealth $i$).

**Question 3.4:** In an LED Christmas light factory, each LED is defective with independent probability 1/500. To test whether lights are defective in a batch, each is connected in sequence, and we attempt to turn on all the lights; if any individual light is defective, no current will run, and the strand will not turn on. Assume we test lights in strands of length 400.

(a) What is the probability that the strand successfully lights, that is, no light is defective?

(b) What is the probability that precisely one light is defective?

(c) What is the probability that the strand will not turn on (that is, at least one light is defective)? Give an exact formula, and then provide answer accurate to a few digits.

Suppose now that the blood test yields a positive result.

(d) What is the probability, under this circumstance, that more than one person has the disease?

(e) Now, suppose we know that the first light is defective. Given this, what is the probability that more than 1 light is defective? Give an exact answer and calculate your result to a few digits.

**Question 3.5**$^\star$ (Gambling with better odds)**:** We revisit Question 3.3 but modify the probabilities a bit. Now at each round of the game, we bet 1, but increase our wealth by 1 with probability $\frac{1+\epsilon}{2}$ and lose 1 with probability $\frac{1-\epsilon}{2}$, where $0 < \epsilon \leq 1$ is some known value. In this new version of the game, let

$$p_i := \mathbb{P}(\text{we hit } m \text{ before hitting } 0 \mid \text{begin with wealth } i).$$

(a) Show that for $i = 1, 2, \ldots, m-1$,

$$p_i = \frac{1+\epsilon}{2} p_{i+1} + \frac{1-\epsilon}{2} p_{i-1}.$$

(b) Noting that $p_m = 1$ and $p_0 = 0$, use the previous result to conclude that

$$\frac{p_{i+1} - p_i}{p_i - p_{i-1}} = \frac{1-\epsilon}{1+\epsilon} \tag{3.1}$$

for $i = 1, 2, \ldots, m-1$.

(c) Define

$$Z_m := \sum_{j=0}^{m-1} \left(\frac{1-\epsilon}{1+\epsilon}\right)^j = \frac{1 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^m}{1 - \frac{1-\epsilon}{1+\epsilon}}.$$

Show that

$$p_i = \frac{1}{Z_m} \sum_{j=0}^{i-1} \left( \frac{1-\epsilon}{1+\epsilon} \right)^j$$

solves the equation (3.1), and so gives the values $p_i$. (Treat $\sum_{j=0}^{-1} a_j = 0$ for any $a_j$, as it is an empty sum.)

(d) Suppose the casino has infinite money (i.e., $m = +\infty$). What is the probability that the gambler never stops gambling (i.e., never runs out of money), assuming they begin with wealth $i = 1$?