Stats 117 Problem Set 3

Due: Monday, April 20 5:00 p.m. on Gradescope

Please show your work for each exercise. If you collaborate with someone else—this is fine—be sure to note that in your homework submission. You must each write up separate answer sets. Any starred exercise is optional: they are extra challenging theoretical exercises for further developing your mastery.

Question 3.1: There are 3 coins in a box. One is a two-headed coin, another is a fair coin, and the third is a weighted coin that comes up heads 75% of the time. When one of the 3 coins is selected at random and flipped, it comes up heads. What is the probability that it was the two-headed coin?

Question 3.2: In college basketball, when a player is fouled while not in the act of shooting, and the opposing team is "in the penalty," the player is awarded a "1 and 1." In the 1 and 1, the player is awarded one free throw, and if that free throw goes in, the player is awarded a second free throw. Find the PMF of Y, the number of points scored in a 1 and 1 assuming that the probability of a free throw going in is 0.7, independent of any other free throw.

Question 3.3 (Gambling in an infinite casino)**:** Suppose we play a game, where at each step of the game, we bet \$1, and with 50% probability increase our wealth by \$1, and otherwise lose \$1. (The game has even odds.) If we hit a bank balance of \$0, we lose the game and stop playing; we also decide that we will stop playing if our balance hits \$$m$ for some finite m. Let

$$p_i = \mathbb{P}(\text{we hit } m \text{ before hitting } 0 \mid \text{start with wealth } i).$$

In this question, we will solve for the probabilities of "winning out" given that we begin with a particular wealth i.

(a) Justify the equality that for $i = 1, 2, \ldots, m-1$,

$$p_i = \frac{1}{2}(p_{i+1} + p_{i-1}) \text{ and so } p_{i+1} - p_i = p_i - p_{i-1}.$$

That is, the probability we win out from wealth i is half $\times$ the probability we win out from wealth $i+1$ plus half $\times$ the probability we win out from wealth $i-1$.

(b) Why do $p_0 = 0$ and $p_m = 1$?

(c) Show that $p_i = \frac{i}{m}$ for each $i = 0, 1, \ldots, m$. (It is enough to verify that this solves the equation in part (a).)

Convince yourself of the *gambler's ruin:* as $m \to \infty$, that is, the point at which we give up and walk away with our wealth, $p_i \to 0$ for any fixed initial wealth i. (That is, no matter what, we eventually lose all our money.)

Question 3.4: Each of 500 soldiers in an army company independently has a certain disease with probability $1/10^3$. This disease will show up in a blood test, and to facilitate matters, blood samples from all 500 soldiers are pooled and tested.[1]

[1] This type of pooling of samples allows faster turnaround time for disease tests: if positivity rates are low, and no one in a group is positive, we can perform a single test to eliminate *everyone* in the group from having the disease.

(a) What is the probability that no individual has the disease?

(b) What is the probability that precisely one individual has the disease?

(c) What is the probability that the blood test will be positive (that is, at least one person has the disease)? Give an exact formula, and then provide answer accurate to a few digits.

Suppose now that the blood test yields a positive result.

(d) What is the probability, under this circumstance, that more than one person has the disease?

Now, suppose one of the 500 people is Jones, who knows that he has the disease.

(e) Given that Jones has the disease, what is the probability that more than 1 person has the disease? Give an exact answer and calculate your result to a few digits.

Question 3.5* (Gambling with better odds)**:** We revisit Question 3.3 but modify the probabilities a bit. Now at each round of the game, we bet 1, but increase our wealth by 1 with probability $\frac{1+\epsilon}{2}$ and lose 1 with probability $\frac{1-\epsilon}{2}$, where $0 < \epsilon \leq 1$ is some known value. In this new version of the game, let

$$p_i := \mathbb{P}(\text{we hit } m \text{ before hitting } 0 \mid \text{begin with wealth } i).$$

(a) Show that for $i = 1, 2, \ldots, m - 1$,

$$p_i = \frac{1+\epsilon}{2} p_{i+1} + \frac{1-\epsilon}{2} p_{i-1}.$$

(b) Noting that $p_m = 1$ and $p_0 = 0$, use the previous result to conclude that

$$\frac{p_{i+1} - p_i}{p_i - p_{i-1}} = \frac{1-\epsilon}{1+\epsilon} \tag{3.1}$$

for $i = 1, 2, \ldots, m - 1$.

(c) Define

$$Z_m := \sum_{j=0}^{m-1} \left(\frac{1-\epsilon}{1+\epsilon}\right)^j = \frac{1 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^m}{1 - \frac{1-\epsilon}{1+\epsilon}}.$$

Show that

$$p_i = \frac{1}{Z_m} \sum_{j=0}^{i-1} \left(\frac{1-\epsilon}{1+\epsilon}\right)^j$$

solves the equation (3.1), and so gives the values p_i. (Treat $\sum_{j=0}^{-1} a_j = 0$ for any a_j, as it is an empty sum.)

(d) Suppose the casino has infinite money (i.e., $m = +\infty$). What is the probability that the gambler never stops gambling (i.e., never runs out of money), assuming they begin with wealth $i = 1$?