Stats 117 Problem Set 8

Due: Tuesday, May 26 5:00 p.m. on Gradescope

Please show your work for each exercise. If you collaborate with someone else—this is fine—be sure to note that in your homework submission. You must each write up separate answer sets. Any starred exercise is optional: they are extra challenging theoretical exercises for further developing your mastery.

Question 8.1:

(a) Let X and Y be independent $\mathsf{Exp}(\lambda)$ random variables, where $\lambda > 0$. Give the distribution of $Z = \min\{X, Y\}$. *Hint.* Use the CDF of Z. It should have a familiar form.

(b) Let $X_1, \ldots, X_k$ be independent $\mathsf{Exp}(\lambda)$ random variables, where $k \in \mathbb{N}$. Give the distribution of $Z = \min\{X_1, \ldots, X_k\}$.

Question 8.2: You inflate a spherical balloon in a single breath. The volume of air you exhale in a single breath (in liters) is a $\mathsf{Uni}[3.375, 8]$ random variable, what is the expected radius of the ballon in centimeters? Remember that a cubic liter is 1000cm^3, and the volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius.

Question 8.3 (Blitzstein and Huang, Ex. 5.23)**:** Alice is trying to transmit the answer to a yes/no questoin to Bob over a noisy channel. She encodes "yes" as 1 and "no" as 0, and sends the appropriate value. However, the channel adds noise; specifically, Bob receives what Alice sends plus a $\mathsf{N}(0, \sigma^2)$ noise term (the noise is independent of what Alice sends). If Bob receives a value greater than $\frac{1}{2}$ he interprets it as "yes"; otherwise, he interprets it as "no".

(a) Find the probability that Bob understands Alice correctly.

(b) What happens to the result from (a) if σ is very small? What about if σ is very large? Explain intuitively why the results in these extreme cases make sense.

Question 8.4: A point is chosen uniformly along the length of a stick, and the stick is broken at the point. What is the probability the left segment is more than twice as long as the right segment? *Hint.* Assume the length of the stick is 1, and let X be the point at which the stick is broken, so that X is the length of the left segment.

Question 8.5: Let X be a continuous random variable with CDF

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3/216 & \text{if } 0 \le x \le 6 \\ 1 & \text{if } x > 6. \end{cases}$$

(a) Give the PDF of X.

(b) Give the expectation $\mathbb{E}[X]$ of X.

(c) Calculate the variance $\text{Var}(X)$ of X.

(d) Calculate the standard deviation of X.

(e) What is the probability that X is within 1 standard deviation of its expected value?

Question 8.6: I used my Geiger counter to count radiation detections in my office for 50 seconds. On average, it provides about 1 radiation event per second, so that the (background) radiation events should have $\mathsf{Exp}(1)$ distribution. In Figure 1 we provide three plots:

i. One is the actual times of the radiation events.

ii. One is a simulation of radiation events (particle detections), assuming the background rate is $\lambda = 2$ events per second.

iii. One is a simulation of radiation events (particle detections), assuming the background rate is $\lambda = \frac{1}{2}$ an event per second.

So in the two fake plots, if $X_i \sim \mathsf{Exp}(\lambda)$ are independent exponential random variables, we plot a tick mark at $T_1 = X_1$, $T_2 = X_1 + X_2$, $T_3 = X_1 + X_2 + X_3$, and so on. Which figure corresponds to each of i, ii, and iii?

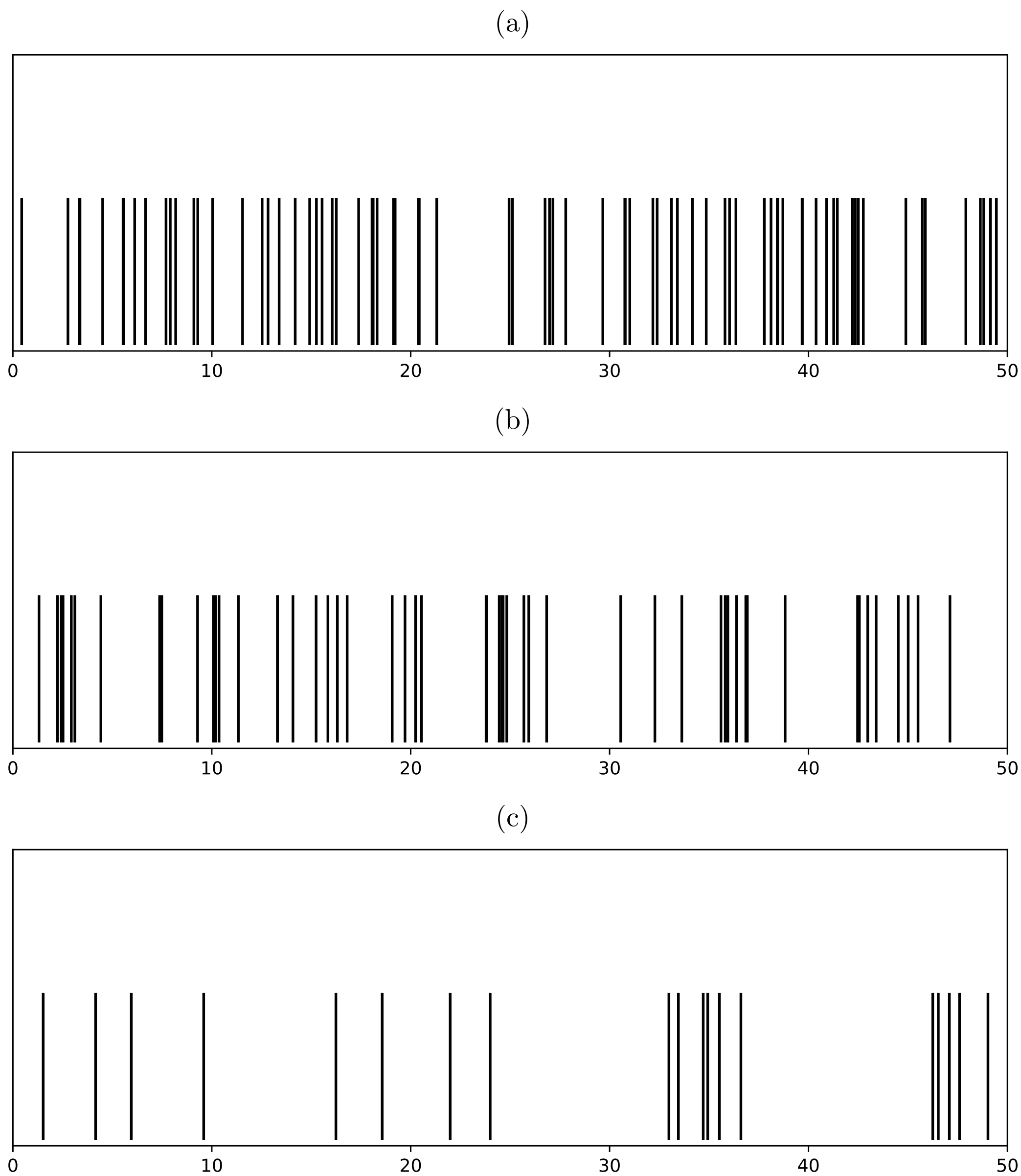

Figure 1. Arrival times for either radiation events or simulated radiation events with $\lambda = \frac{1}{2}$, $\lambda = 1$, or $\lambda = 2$. Horizontal axis is time, vertical ticks indicate particle arrival.